VII. *On the Dispersion in Artificial Double Refraction.*

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Received January 25,—Read February 28, 1907.

Summary of Contents.

§ 1. *Introduction.*

It is well known that glass and other transparent isotropic substances, when compressed unequally in different directions, behave like doubly-refracting substances and exhibit the colours of polarized light. Attention was first called to this by Fresnel ('Annales de Chimie et de Physique,' vol. XX.), and by Sir David Brewster ('Phil. Trans.,' 1816). For further investigations in this field, reference may be made to F. E. Neumann ('Abhandlungen der k. Acad. v. Wissenschaften zu Berlin,' 1841, II.; see also 'Pogg. Ann.,' vol. LIV.); to Clerk Maxwell ('Trans. Roy. Soc. Edin.,' vol. XX., Part I.; or 'Collected Papers,' vol. I.); to G. Wertheim ('Annales de Chimie et de Physique,' ser. 3, vol. XL., p. 156); to J. Kerr ('Phil. Mag.,' October, 1888); and to F. Pockels ("Über die Änderung des optischen Verhaltens verschiedener Gläser durch elastische Deformation," 'Ann. d. Physik,' 1902, ser. IV., vol. VII., p. 745). Of these only Wertheim and Pockels have considered how the effect varies with the nature of the light employed.

If homogeneous parallel light is passed perpendicularly through a plate of thickness τ which is subjected to principal stresses P, Q in its plane, these stresses being uniform throughout, then it is found that the light on traversing the plate is broken up into two rays polarized in the directions of principal stress. The relative retardation in centimetres of these rays on emergence is given by

$$R = (\mu_1 - \mu_2)\,\tau,$$

where μ_1, μ_2 are the indices of refraction of the two rays.

Now experiments have shown that $\mu_1 - \mu_2$ is very approximately proportional to the principal stress difference in the wave-front, $P-Q$. Whether this is true for high values of $P-Q$ is not certain, and some experiments to be described in the following pages (see § 19) will show that the proportionality of $\mu_1-\mu_2$ to $P-Q$ in all cases must still be regarded as doubtful. Assuming, however, this law, which is certainly very nearly true in most cases, at all events when P, Q are stresses of the same type (tensions, or pressures), we have

$$R = C\,(P-Q)\,\tau,$$

where C is a coefficient depending only on the nature of the material and on the wave-length of the light used. This coefficient C will be spoken of in what follows as the "stress-optical coefficient."

Wertheim, from observations of a uniformly compressed block of glass through which he passed successively (i.) sodium light, (ii.) white light, (iii.) white light filtered through a red glass, stated the following law:—

The relative retardation in air is constant for all colours. In other words, the stress-optical coefficient C is independent of the wave-length; the difference of the refractive indices is therefore likewise independent of the wave-length, that is, the double refraction due to elastic strain exhibits *no dispersion.*

POCKELS, in his more recent investigation, observed the artificial double refraction in a number of Jena glasses. His observations, though not primarily intended to test the effect of colour, nevertheless gave exceedingly valuable results in this connection, insomuch as POCKELS experimented with three different kinds of nearly homogeneous light, namely, those of sodium, lithium, and thallium. The results would therefore be far more precise than those obtained with very mixed colours by WERTHEIM. They show that, in certain glasses, the stress-optical coefficient does vary with the wave-length, being numerically greater in the green than in the red; and in very heavy lead glasses this variation is more rapid as we approach the blue end of the spectrum.

Some years ago the present author, being at the time unaware of POCKELS' experiments, devised a method of observing the variation of the coefficient C *continuously* throughout the spectrum, the object being to test the exactness of WERTHEIM'S law.

An account of this method, which was modified and improved from time to time, and of the experiments undertaken to carry it out, will be found in the 'Camb. Phil. Soc. Proc.,' vol. XI., Part VI.; vol. XII., Part I.; vol. XII., Part V. These experiments amply confirmed the results of POCKELS. They also showed that the chief desideratum for obtaining accurate results was that the stress in the glass slab through which the light was passed should be sufficiently uniform. Now the compression apparatus which was used by the author, and by previous experimenters, suffered from the defect that it was practically impossible to adjust it so as to obtain a uniform pressure in the slab of glass under observation. Moreover, what adjustment could be made was long and difficult, and could be attained only by trial; it appeared further that this adjustment was disturbed, in a way that could not be calculated and allowed for, when the load was altered. This greatly reduced the accuracy expected.

An apparatus was then devised, with a view to obtaining a system in which the stresses should be known exactly and in which the optical effects should be the same as those due to uniform pressure in a slab of constant thickness. For the purposes of this research a Government Grant was kindly placed at the disposal of the author by the Royal Society, whereby the necessary apparatus could be constructed and the expensive glasses required for the research purchased. The present paper is an account of the experiments carried out with the new apparatus and of the results reached.

PART I.

THEORY OF THE EXPERIMENT AND DISCUSSION OF THE VARIOUS ERRORS.

§ 2. *Simple Theory of the Experiment.*

Let N, F (fig. 1) be two rectangular slabs of glass, whose cross-sections are shown in the figure. The slabs are bent in a vertical plane by couples without shear, whose axes are horizontal and parallel to the plane of the cross-section. How such couples are obtained will be explained subsequently.

The horizontal and vertical sides of the cross-sections of N and F are $(2a_1, 2b_1)$, $(2a_2, 2b_2)$ respectively, and the centres of the two cross-sections are O_1 and O_2. Let S be a point-source of light; S′ its image after passing through N; P_1, P_2 the points in which any ray through S meets the vertical midplanes of N, F respectively.

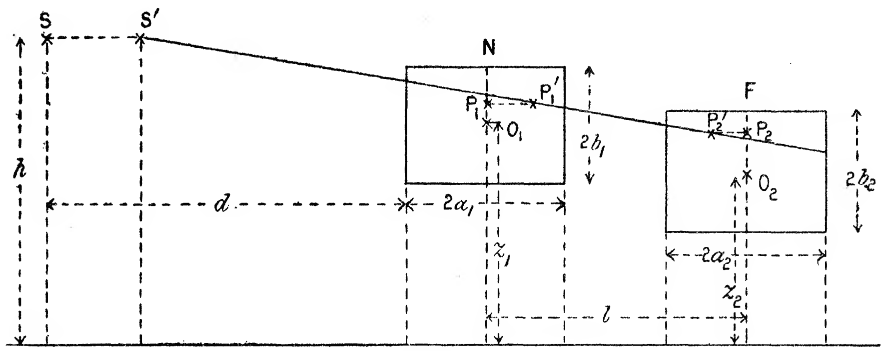

Fig. 1.

Let P'_1 be the image of P_1, after a single refraction at the inner face of N (the one towards F), and P'_2 the image of P_2 after a single refraction at the inner face of F (the one towards N).

Then it is evident that S′, P'_1, P'_2 must be in one straight line.

Let

h = height of S above a fixed horizontal plane,
z_1 = ,, O_1 ,, the same plane,
z_2 = ,, O_2 ,, ,, ,,
y_1 = ,, P_1 ,, O_1,
y_2 = ,, P_2 ,, O_2,
d = distance of S from the nearer face of N,
l = ,, between midplanes of N, F,
μ_1, μ_2 = refractive indices of N, F respectively.

We have

$$SS' = 2a_1(\mu_1-1)/\mu_1, \quad P_1P'_1 = a_1(\mu_1-1)/\mu_1, \quad P_2P'_2 = a_2(\mu_2-1)/\mu_2,$$

and since S′, P_1', P_2' are collinear

$$\frac{z_2+y_2-(z_1+y_1)}{l-a_1-a_2+a_1/\mu_1+a_2/\mu_2} = \frac{z_1+y_1-h}{d+a_1/\mu_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

or, writing

$$l-a_1-a_2+a_1/\mu_1+a_2/\mu_2 = \sigma(d+a_1/\mu_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

(1) becomes

$$z_2+y_2 = (z_1+y_1)(1+\sigma) - \sigma h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

Suppose now monochromatic light proceeds from S.

Let C_1, C_2 be the stress-optical coefficients of the two slabs for this kind of light, M_1, M_2 their bending moments reckoned positive when the slabs are bent concave downwards.

The relative retardation after traversing the first slab for any ray which passes through P_1 at a small angle to the horizontal $= 2a_1 . C_1 (3M_1 y_1/4a_1 b_1^3)$.

For, although the stress is not uniform along the path of the ray, the mean stress along the ray = stress at the middle point, since the stress varies linearly as the distance from the neutral axis. Also the length of the ray differs from the breadth of the slab only by quantities of the second order. Hence the result above.

Accordingly the total relative retardation, after passing through the two slabs, is given by

$$R = (3M_1/2b_1^3) C_1 y_1 + (3M_2/2b_2^3) C_2 y_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

Substituting for y_2 from (3) into (4),

$$R = (3M_1/2b_1^3) C_1 y_1 - (3M_2/2b_2^3) C_2 z_2 + (3M_2/2b_2^3) C_2 [(z_1+y_1)(1+\sigma) - \sigma h] \qquad (5).$$

Now, if R is to be independent of y_1, we must have

$$3M_1 C_1/2b_1^3 + 3M_2 C_2 (1+\sigma)/2b_2^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

This condition will not of course be accurately fulfilled for all colours at the same time; in the first place, because C_1 and C_2 do not in general follow the same law of variation for the two slabs; in the second place, because σ contains μ, and therefore involves the wave-length.

It is easy to see that the latter cause of error is quite negligible. For if d be large compared with a_1 or a_2, which was the case in all the experiments, the error introduced in σ by a variation $\delta\mu$ in the refractive index (taking $\mu_1 = \mu_2$, which is practically true) is approximately

$$(a_1 + a_2)\, \delta\mu/\mu^2 d.$$

Now, $a_1 + a_2 = 3$ centims. in the experiments to be described; $d = 250$ centims. about (or larger), $\delta\mu = 0{\cdot}01$ between the C and F lines of the spectrum which represent fairly well the extreme range of the observations.

Hence, μ being about $1{\cdot}5$,

$$(a_1 + a_2)\, \delta\mu/\mu^2 d = {\cdot}01/187{\cdot}5 < {\cdot}00006.$$

Accordingly the error introduced by this cause would correspond to an error in C_2 of less than 6 in a hundred thousand, an error which is absolutely negligible, since the errors of ordinary observations in the method to be described amount to $\frac{1}{400}$ or $\frac{1}{500}$ of C. (See 'Camb. Phil. Soc. Proc.,' vol. XII., p. 58.)

The different variation of C_1 and C_2 with the wave-length would be far more important. This, however, need not be considered, for the two slabs N and F are taken from the same cast, so far as possible, so that C_1 and C_2 should be identical. In some cases it was found that C_1 and C_2 differed; but, at the same time, the

experiments showed conclusively that for slabs of the same material C_1 and C_2 remained very approximately proportional one to the other for all the values of λ examined. In this way condition (6) is satisfied independently of the wave-length.

It follows from (5), using (6), that the relative retardation is given by

$$R = (3M_2C_2/2b_2^3)[z_1 - z_2 + \sigma(z_1 - h)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Accordingly the relative retardation, after the light from such a point-source has traversed the two slabs, is the same for all the rays from S. Two such slabs are therefore optically equivalent to a single slab which would be under an accurately uniform tension.

By properly adjusting the differences of height, $z_1 - z_2$, $z_1 - h$, the amount of relative retardation may be adjusted within certain limits.

In general, l will be chosen small with respect to d. Thus σ is a proper fraction, say of order $\frac{1}{10}$. Equation (6) thus shows that M_1 and M_2 are to be chosen of opposite signs, and approximately equal in magnitude.

This gives at once the physical explanation of the result (6). The rays pass through approximately horizontally. If we compare two rays passing through at different levels, the ray which passes through the regions of greater tension in N passes through the regions of lesser tension (or greater pressure) in F, and the two variations balance one another.

Further, since the amount of relative retardation as given by (7) involves only the relative heights of the axes of the slabs and the source of light, the latter may be moved parallel to the axes of the slabs without affecting the relative retardation. Hence a horizontal line-source, parallel to the axes of the slabs, may be used instead of a point-source. This was, in fact, indispensable in order to obtain the required intensity.

§ 3. *Description of the Apparatus.*

Light from an arc lamp L was passed through a condensing lens C and through a thin horizontal slit T (fig. 2), which was placed from $2\frac{1}{2}$ to 3 metres away from the

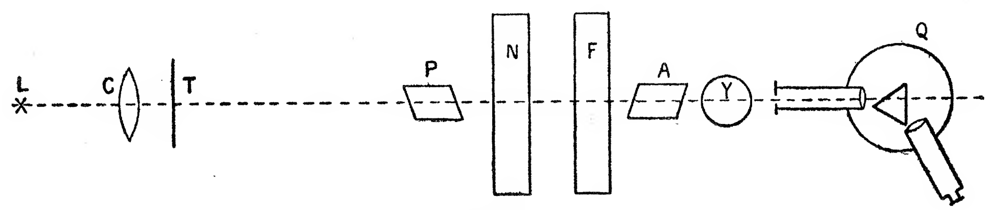

Fig. 2.

glasses and straining apparatus. It was polarized by a Nicol P, whose polarizing plane was roughly at 45° to the horizontal, and then passed through the two slabs N and F. These were adjusted so that their levels differed very nearly by $\frac{1}{2}$ centim.

The two slabs were cut from the same piece of glass, and every precaution was taken to ensure that they should be as nearly identical as possible. The dimensions of the cross-sections were practically the same, namely, in the notation of the last section, $2a_1 = 2a_2 = 3$ centims. and $2b_1 = 2b_2 = 1$ centim.

The length of each slab was about 13 centims.

Bending moments of opposite sign, in a vertical plane, were applied to these, so that the light passed through parts of the glass either altogether under tension, or altogether under pressure, according to the manner in which the bending moment was applied. Of the method of applying such bending moment a fuller account will be given below.

After emerging from the two beams the pencil of light traversed a Nicol A, which was crossed with the Nicol P. It was then focussed by a cylindrical lens Y (which consisted in practice of a glass beaker filled with water) upon the vertical slit of a spectroscope Q and the spectrum observed in the usual way.

The condensing lens C was focussed approximately upon the Nicol P; both C and Y were introduced in order to improve the illumination. It was found otherwise that so much light was lost that only a very faint spectrum could be obtained, and this was useless for the purpose of the observations.

The latter consisted in measuring accurately the position in the spectrum of the black bands corresponding to light completely quenched between the Nicols P and A. Light of any colour will, of course, be quenched between crossed Nicols when the relative retardation of the two rays (polarized in horizontal and vertical planes respectively) in which it is split up by the strained glass amounts to an integral multiple of the wave-length.

Referring to formula (7) this occurs when

$$n\lambda = (3M_2C_2/2b_2^3)[z_1 - z_2 + \sigma(z_1 - h)] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

n being an integer. If C_2 were independent of the wave-length, as WERTHEIM's law would require, then, for a band of a given *order*, n is fixed and the wave-length λ of extinction is proportional to M_2.

If, however, C_2 varies with λ, then λ/C_2 is proportional to M_2.

By observing the values of λ corresponding to a given M_2, and varying M_2, we obtain the relative magnitude of the coefficient C_2 for these varying values of λ.

The probable error of setting on the centre of a black band was calculated by the author in the 'Camb. Phil. Soc. Proc.,' vol. XII., Pt. V., pp. 314–315, and was found to be about 1′, so that the wave-length of extinction is determined with a proportional error smaller than 0·002. The average error due to inaccuracy in setting the cross-wires in the eye-piece upon the centre of the black band is then about 8 to 10 tenth-metres, so that the wave-lengths may be considered known accurately to three figures.

The bending moments were applied to the slabs by means of the apparatus shown

in fig. 3. The slab G rested on two knife-edges R and S. On it rested two other knife-edges U and V, supporting a graduated steel bar I. Fixed to the top of I was a triangular knife-edge K, from the projecting extremities of which hung two

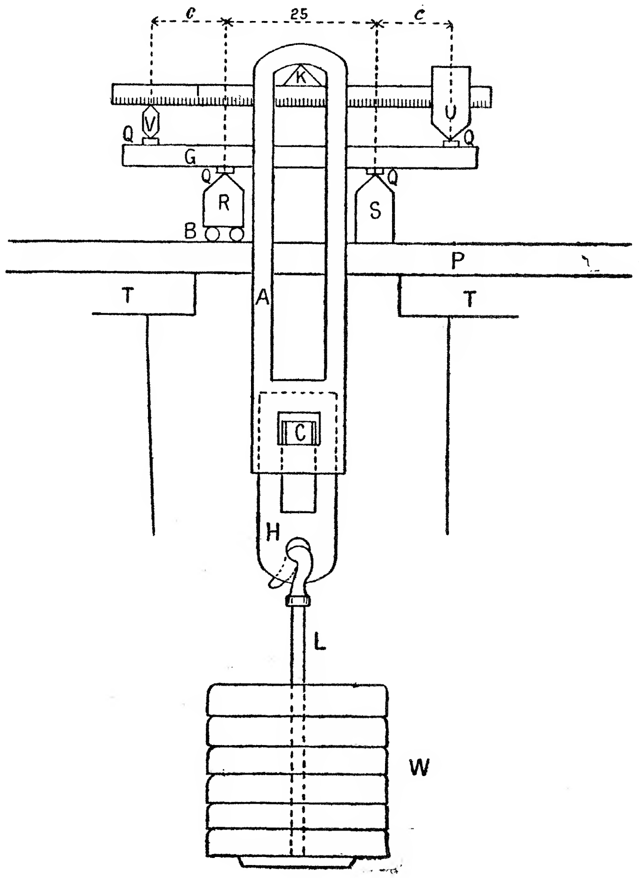

Fig. 3.

symmetrical hangers A. These passed through holes cut in the bed-plate P, which supported the whole apparatus, and by means of a cross-piece C and another hanger H a load W was applied which acted on I vertically downwards at its middle point.

In order to ensure that the reactions between G and the knife-edges R, V should be vertical, the knife-edge R rested on steel bicycle balls B, so that it would readily move under horizontal friction; V was made a double knife-edge, the plane containing the two edges being carefully adjusted to be vertical. The knife-edge U could slide along I and be clamped in any desired position. When U was clamped and the load applied, the apparatus was perfectly stable, the knife-edge S being kept in its place by the friction of the bed-plate. In order to prevent the knife-edges from cutting into the glass and breaking it under the large loads applied, four small slips of steel Q were inserted between the knife-edges and the glass. These distributed the actual stress without altering the actual statical resultants, and at points near the centre of

the slab the effect of such local perturbations must be negligible. (See 'Phil. Trans.,' A, vol. 201, pp. 114, 145.)

When the apparatus is in perfect adjustment K is exactly midway (measured horizontally) between U and V, and the horizontal distances between the edges of V and R and of U and S are equal. If these be each c, then the bending moment applied to the part of the slab between R and S is constant and equal to $\frac{1}{2}c$W.

For, since the reaction at the upper edge of V is vertical and the load at K is vertical, then the reaction at the lower edge of U is shown to be vertical by considering U, I, K as one system. Thus the reactions at the lower edge of U and V are each equal to $\frac{1}{2}$W. Again, the reaction at the upper edge of R is vertical and therefore the reaction at the upper edge of S is also vertical. Hence these reactions also are equal to $\frac{1}{2}$W.

Also it is to be noted that, if the adjustment be perfect, the bending moment applied to the beam or slab is a *pure* bending moment. There is no total shear across any cross-section between R and S.

In such a case it is well known that the distribution of stress obeys accurately the Euler-Bernouilli laws and consists only of a tension $My/\Delta k^2$ parallel to the axis of the beam, where M = applied bending moment, y = height above neutral axis (horizontal line drawn through the centroid of the cross-section in the plane of the cross-section), Δ = area of cross-section, k = its radius of gyration about neutral axis. The formulæ (4) and (7) are therefore verified.

In fig. 3 the knife-edges U and V are outside R and S. The bending moment is therefore positive, with the convention of p. 266. For the second beam the arrangement is the same, except that now U and V are inside R and S, so that the bending moment is negative.

The difference of height between the slabs was obtained by placing the knife-edges R, S for one of the beams upon two steel blocks of height 0·5 centim. instead of directly upon the bed-plate. The bed-plate itself was a solid plate of steel, very strong and resting upon two heavy tables T of the same height.

In the above description no account has been taken of a large number of small errors which must theoretically affect the method.

The principal are the following:—(1) In the theory explained in § 2 modifications will be introduced owing to the fact that a polarizing Nicol is introduced in the path of the rays of light between the source and the slabs. (2) The source of light is not a line-source, but a slit of finite breadth. (3) When the load is applied, the middle part of one beam rises and the other sinks: thus the heights z_1, z_2 and the relative height z_1-z_2 in formula (7) are not fixed. (4) The bed-plate P and the tables T are not absolutely rigid. This will alter z_1 and z_2, but not z_1-z_2. (5) The rays do not go through the glass horizontally and at right angles to the axes of the slabs, and the assumption that the mean retardation = retardation at mid-point of path is only an approximation. (6) The slit used as a source of light is not accurately horizontal.

(7) The knife-edges are never quite accurately adjusted. (8) The weight of the beams themselves will affect the stresses. (9) The beams are not always perfectly annealed and the permanent stresses in the glass modify the appearances.

In the following sections the corrections due to these errors will be investigated.

§ 4. *Effect of Introducing the Polarizing Nicol.*

We shall now consider the effect of introducing the polarizing Nicol upon the inclination of the rays of light. In order to estimate the magnitude of this effect, it will be sufficient to treat the Nicol as a singly-refracting substance. If the larger index of refraction be adopted this should, in general, give us an upper limit to the error introduced. If no sensible disturbance is found to be thus introduced, we may assume that this will be the case in the actual experiment.

Let S (fig. 4) be the source of light, P the image of a point of the mid-plane of the

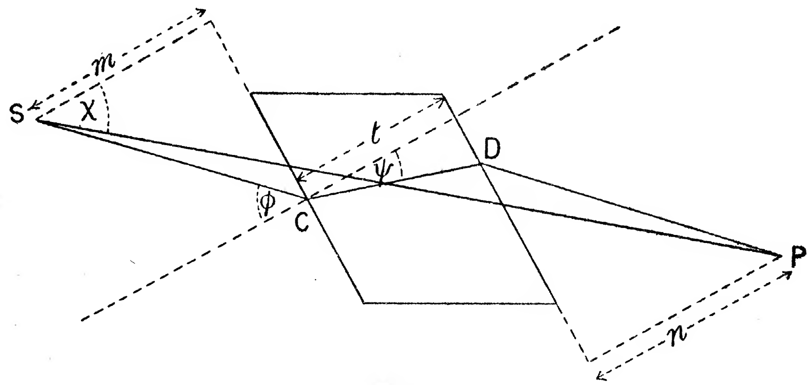

Fig. 4.

nearer slab, viewed by refraction through the face nearest S. If the Nicol were not present the light would travel along the line SP. In consequence of the introduction of the Nicol it travels along the broken path SCDP.

Let ϕ, ψ be the angles of incidence and refraction, χ the angle which SP makes with the normal to the faces of the Nicol.

Let the perpendicular distances of S and P from the nearer faces of the Nicol be m and n, and the thickness of the Nicol be t.

Then

$$(m+n)\tan\phi+t\tan\psi = (m+n+t)\tan\chi,$$

or, writing $t/(m+n+t) = \gamma$,

$$\tan\chi-\tan\phi = \gamma(\tan\psi-\tan\phi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

Using $\mu\sin\psi = \sin\phi$,

$$\tan\psi = \tan\phi\,[\mu^2+(\mu^2-1)\tan^2\phi]^{-1/2}.$$

Hence (9) becomes, retaining only first powers of γ,

$$\tan\chi-\tan\phi = \gamma\tan\chi\,([\mu^2+(\mu^2-1)\tan^2\chi]^{-1/2}-1) \quad . \quad . \quad . \quad . \quad (10).$$

Now in the experiments $\gamma < \frac{1}{50}$ and the maximum variation in χ for rays passing through the slabs amounts only to $\frac{1}{500}$.

But from (10)

$$\frac{d}{d\chi}(\tan\chi - \tan\phi) = \gamma \sec^2\chi \{\mu^2(\mu^2 + [\mu^2-1]\tan^2\chi)^{-3/2} - 1\} \quad . \quad . \quad . \quad (11).$$

Accordingly the greatest variation in $\tan\chi - \tan\phi$ for the rays passing through the slabs is less than

$$(\tfrac{1}{25000})\sec^2\chi\{\mu^2(\mu^2 + [\mu^2-1]\tan^2\chi)^{-3/2} - 1\} \quad . \quad . \quad . \quad . \quad . \quad (12).$$

To compute the order of this expression take $\mu = 1{\cdot}5$ and $\chi = 30°$, which last is an extreme estimate. (12) gives

$$\delta(\tan\chi - \tan\phi) = -0{\cdot}000033,$$

that is,

$$\delta(\chi - \phi) = -0{\cdot}000033 \cos^2\chi = -0{\cdot}000025 \text{ about.}$$

The change in relative level of the points P_1, P_2 of fig. 1 due to the above amounts to $(0{\cdot}000025)\,l$, and in the experiments $l = 18$ centims. roughly. Thus the change in relative level $= 0{\cdot}00045$ centim. The proportional change in the total effective stress = change of relative level ÷ semi-height of slab $= 0{\cdot}0009$, and this will produce a negligible error in the stress-optical coefficient. Accordingly for monochromatic light the effect is to increase the obliquity of all rays by a small constant amount, or to change the effective height h of the slit. As the absolute value of h is not known, and will be found not to enter into the calculations, the presence of the Nicol will not affect the observations for monochromatic light.

For white light, however, it may do so if the quantity $\tan\phi - \tan\chi$ vary sensibly with the colour of the light used, other things remaining the same.

From (10)

$$\delta(\tan\chi - \tan\phi) = -\mu\gamma\delta\mu \sin\chi(\mu^2 - \sin^2\chi)^{-3/2}.$$

Now for calcite

$$\lambda = 6708, \qquad \mu_e = 1{\cdot}484, \qquad \mu_0 = 1{\cdot}653,$$

$$\lambda = 5350, \qquad \mu_e = 1{\cdot}488, \qquad \mu_0 = 1{\cdot}658,$$

$$\delta\mu_e = 0{\cdot}004, \qquad \delta\mu_0 = 0{\cdot}005.$$

Taking the ordinary index as the basis of computation and $\chi = 30°$ as before,

$$\delta(\tan\chi - \tan\phi) = -0{\cdot}000026 \text{ nearly.}$$

The proportional error in C deduced from this is obtained by multiplying by $l\cos^2\chi/b$, *i.e.*, by 27. It is therefore $0{\cdot}00070$. This error corresponds only to the

dispersion between the lithium and thallium lines. The error will be greater when we reach the violet end of the spectrum, but it will still be too small to affect the observations.

In experiments demanding great accuracy it might be desirable to polarize the light before it passes through the slit. The accuracy possible under the present circumstances did not seem to justify this additional complication in the apparatus.

§ 5. *Effect of Finite Breadth of the Source.*

It may be shown that if the slit have a finite breadth $2e$, the intensity of the light that gets through is proportional to

$$1 - \frac{\sin g\theta}{g\theta} \cos \theta,$$

where

$$\theta \doteq 2\pi R_0/\lambda, \qquad g = \sigma e/[z_1 - z_2 + \sigma (z_1 - h)],$$

R_0 being the relative retardation corresponding to the middle of the slit.

In the actual case $g = \frac{1}{150}$ approximately.

The minima are given approximately by

$$\theta = 2n\pi (1 - \tfrac{1}{3}g^2).$$

The proportional error in the wave-length of extinction is therefore $\frac{1}{67500}$, which is negligible.

Also the minimum no longer corresponds theoretically to perfect darkness, but with a slit between $\frac{1}{2}$ millim. and 1 millim. wide the bands were very dark and quite definite.

§ 6. *Effect of Relative Rise and Fall of the Two Beams and of Elastic Yielding of the Bed-plate and Knife-edges.*

Owing to the elasticity of the glass, the middle parts of the beams will undergo a vertical shift owing to flexure, and the bed-plate and apparatus as a whole will sink.

In consequence we have variations δz_1, δz_2, δh depending on the applied load.

Thus the right-hand side of equation (8) is multiplied by a factor

$$1 + [\delta z_1 - \delta z_2 + \sigma (\delta z_1 - \delta h)]/[z_1 - z_2 + \sigma (z_1 - h)].$$

This may be allowed for by supposing M_2 (or W) multiplied by the same factor, equation (8) remaining otherwise unaltered.

The effect is then to add to the applied weight a correction

$$W [\delta z_1 - \delta z_2 + \sigma (\delta z_1 - \delta h)]/[z_1 - z_2 + \sigma (z_1 - h)].$$

Now the relative rise and fall of the beams themselves is an elastic effect and may be taken, in such a small correction, strictly proportional to the load.

The sinking of the bed-plate was measured experimentally and found to be elastic in its nature, the recovery being complete.

Generally the experiments showed no trace of permanent set, the readings being the same when unloading as when loading.

We may safely assume therefore that $\delta z_1 - \delta z_2 + \sigma(\delta z_1 - \delta h)$ is proportional to W, so that the correction to be applied to W on account of these errors is of the form

$$KW^2.$$

The value of K is uncertain and depends very largely on the circumstances of each experiment.

Using Everett's and Amagat's values of Young's modulus for glass (*i.e.*, between 600,000 and 700,000 kilogs.-weight per square centimetre), the part of K due to relative rise and fall of the two beams was calculated to be about 0·0004. Thus for W = 50 kilogs. the proportional correction is as high as 2 per cent.

The part of K due to the sinking of the bed-plate was found experimentally to be of order 0·00026. Also the experiments could be arranged in such a manner that the two corrections operated in different senses; and this precaution was always taken. Their combined effect will give K of order 0·0001, and even for the highest loads used the correction will be small.

In practice this correction KW^2 was determined from the observations themselves, in a manner explained in § 17. For most sets of observations it was found to be insensible.

§ 7. *Influence of Obliquity on Relative Retardation.*

We may consider the glass as optically made up of a series of horizontal homogeneous layers. In passing from one of these layers to another, the refraction takes place approximately in a plane perpendicular to the optic axis.

It will be sufficient for our purpose to consider a ray passing through in a cross-section, that is, in a plane throughout perpendicular to the optic axis. If the curvature of such a ray be negligible, we may take it that we can neglect the curvature of all oblique rays.

Now if μ be the index of refraction at a point in the glass distant y from the neutral plane,

$$\mu = \mu_0 + \mu_1 y,$$

μ_0 being the index of refraction for the unstrained glass. It may then be proved that the curvature of a ray passing through nearly horizontally is approximately μ_1/μ_0.

Now it has been shown by Kerr ('Phil. Mag.,' October, 1888), and by Pockels ('Ann. d. Physik,' 1902, p. 745), that the absolute variation due to stress in the index of refraction for either ray is of the same order as the difference in the two indices due to the same cause. In general, for the highest stress employed, the

$(\mu_{ex}-\mu_{or})$-gradient is of the order 10^{-4}. Taking $\mu_0 = \frac{3}{2}$, the curvature is of order $\frac{2}{3}10^{-4}$. Hence the greatest deviation from the straight line $= \frac{1}{2}$ (curvature) (thickness of slab)^2 $= 3 \times 10^{-4}$—a divergence which cannot possibly affect the results.

Thus we are justified in treating the paths of the rays as linear. Moreover, the divergence of the ordinary and extraordinary rays after refraction at entrance $= \mu^{-2}(\mu_{ex}-\mu_{or})$ (angle of incidence) very nearly, and it is easily verified that the effect of this is also entirely negligible. Therefore we may treat the two rays as geometrically coincident.

The paths of the rays being linear, the planes of polarization are fixed throughout. For these can be proved to be the plane through the ray, and the line of strain and the plane through the ray perpendicular to the first plane. And the line of strain is always parallel to the axis of the slab.

Also if β = angle between ray and line of strain, the relative retardation introduced by an element ds of path is

$$d\mathrm{R} = \mathrm{CT}\sin^2\beta\, ds.$$

Hence the total relative retardation is

$$\mathrm{R} = 2a\mathrm{CT}_0 \sin^2\beta \sec\gamma,$$

where

T_0 = tension at mid-point of path,

$2a$ = thickness of slab,

γ = inclination of ray to the horizontal perpendicular to the axis.

In practice the limits for $\cos\beta$ are $\pm 0{\cdot}01$, and for γ are $\pm 0{\cdot}02$. It follows that the factor $\sin^2\beta \sec\gamma$ introduces a proportional error less than 10^{-4} in the relative retardation. It may therefore be altogether neglected.

§ 8. *Combined Effect of Flexure and Obliquity.*

The relative vertical displacement of the two slabs due to flexure varies with the cross-section taken. Now the pencil of rays used passes through a comparatively large region of the beams, extending to about 1 centim. on either side of the central cross-section. It is readily shown that the changes in z_1, z_2, due to flexure as we pass from the central cross-section to sections distant x from the central one are given by

$$\left.\begin{aligned}\delta z_1 &= -\,3c_1x^2\mathrm{W}/16\mathrm{E}_1a_1b_1^3 = -3\mathrm{W}/\mathrm{E}\\ \delta z_2 &= +\,3c_2x^2\mathrm{W}/16\mathrm{E}_2a_2b_2^3 = +3\mathrm{W}/\mathrm{E}\end{aligned}\right\}\text{if } x = 1,$$

the slab N being bent concave downwards and F concave upwards.

The greatest possible change in $\delta z_1 - \delta z_2$, due to this cause, is numerically equal to $6\mathrm{W}/\mathrm{E}$ or (taking $\mathrm{E} = 600{,}000$ kilogs.-weight per square centimetre) $= \mathrm{W} \times 10^{-5}$.

The proportional correction in the stress amounts to $10^{-5}\,\mathrm{W}/(z_1-z_2)$ nearly, *i.e.*, to

$10^{-5}.2W$. Thus for the extreme load of 50 kilogs. it is only 10^{-3}, and may be disregarded.

§ 9. *Imperfect Adjustment of the Inclination of the Slit.*

If the slit be not horizontal its different parts will act as different sources of light at different heights h.

It is clear that if the inclination be too great the different parts of the slit will give different dark bands in the spectrum, all overlapping. The integral band will be diffuse in consequence and not readily measureable. It is quite easy, however, to make this adjustment to a nicety, as follows :—

Let AB (fig. 5) be the slit, A′B′ the image of AB in the cylindrical lens (Y of fig. 2) for rays proceeding in a horizontal plane. Then each element P of the slit gives a vertical line of light through P′. Let S_1S_2 be the opening of the slit of the spectroscope. The latter is a good deal smaller than the image A′B′, so that in practice only a moderate length of the luminous slit is used. If now the cylindrical lens be moved to one side or the other, so that S_1S_2 travels from one end of A′B′ to the other, then, if the luminous slit be not horizontal, the band will shift in the spectrum in consequence. When no such shift occurs, we know that the adjustment is very exact. There is very little difficulty in making this adjustment, and accordingly there is no reason for anticipating any sensible error from this cause.

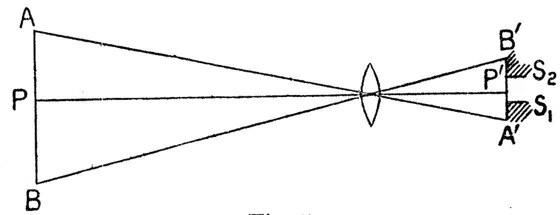

Fig. 5.

§ 10. *Imperfect Horizontal Adjustment of the Knife-edges.*

In practice it is impossible to ensure that the two pairs of knife-edges shall be exactly symmetrical with regard to the vertical through the load. Failure to satisfy this condition introduces shearing stresses in the beams, so that the axes of polarization are no longer horizontal and vertical and further the bending moment varies from cross-section to cross-section.

The complete analytical investigation of the correction in this case is long and difficult, but the results may be summed up as follows, for the simplest case, when only one of the slabs is supposed imperfectly adjusted.

In general there is no longer perfect extinction, so that the band is not quite black.

Assuming that the "overlap" of the two slabs is half their height, the position of the band for rays passing through the edge of either slab is unaltered.

The position of the band for a ray passing exactly at mid-level is shifted towards the *red* end of the spectrum by an amount not exceeding 0·6 of a tenth-metre.

Thus, remembering that rays which have passed through the glass at different

levels correspond to parts of the spectrum also at different levels, we see that the band is no longer straight and vertical, but curved, the convexity being towards the red. This convexity is, however, so small that it would not in any case be observable. If the condition (6) of § 2 is not exactly satisfied the band will still be straight, to a first approximation, but no longer vertical. Thus when the bending moment varies from cross-section to cross-section for light passing to the right of the mid-section the band is tilted one way, for light passing to the left it is tilted the reverse way. The integral effect will be that the thickness of the band will no longer be uniform, but the band is still symmetrical with regard to a vertical line, corresponding to light going through the mid-section. The settings which are made on the *middle* of the band are therefore unaffected.

§ 11. *Imperfect Vertical Adjustment of the Knife-edges.*

It will also happen that the knife-edges will not all be at exactly the same height, so that the axes of the two slabs are not exactly horizontal and parallel. The effect will be that for rays passing through in a plane distant x from the central section

$$z_1 - z_2 + \sigma(z_1 - h) = \mathrm{A} + \mathrm{B}x$$

instead of being exactly constant, B being a small coefficient.

This will broaden the band and render it more diffuse, but will not shift its centre. Observation shows that this effect must be very small, as, in general, the band is very well defined.

§ 12. *Error due to Weight of Beams themselves.*

In computing the stresses no heed has, so far, been paid to the fact that the weights of the glasses themselves will introduce certain stresses in the slabs. The weight of each slab is on the average 120 grammes. This, although very small compared with the total load in most cases, may introduce a small error in the case of the band of the first order, which corresponds to a smaller load.

For the beam N the weight of the glass was found to introduce practically no bending moment in the centre, as the supports were very nearly at the quarter and three-quarter span points.

For the beam F the moment introduced is the same as if the weight on this slab were increased by exactly its own weight.

It is quite easy in practice to eliminate this by adding a small counterpoise to the weight on N.

§ 13. *Error due to Imperfect Annealing.*

We now come to the only error—with the exception of that due to rise and fall—which is sufficiently important to be allowed for in the reduction of the observations.

The annealing of the glasses used, which were supplied by Messrs. ZEISS of Jena, was found to be by no means perfect. In some cases this was revealed even by a cursory inspection between crossed Nicols. In other cases, the glasses being unloaded, a one-wave plate of selenite was introduced between the Nicols, its axes being horizontal and vertical. This showed a black band, on the same principle that the strained glass shows such a band.

Now if the glasses had no residual stress the relative retardations should be the same when the azimuth of the axes of polarization of the selenite plate is altered by 90°.

If there be residual stress, however, it will affect the light differently in these two cases and the band will be shifted. In most cases the existence of such a residual stress was exhibited very plainly by this method. As a rule the band due to the selenite plate was straight and vertical, showing that the residual stress was fairly constant.

If ΔT, ΔU, ΔS be the three components of residual stress in a vertical plane parallel to the axis of the beam, then the axes of polarization make an angle ϕ with the horizontal, where

$$\tan 2\phi = 2\Delta S/(T+\Delta T-\Delta U)$$

and the principal stress difference

$$P-Q = \sqrt{(T+\Delta T-\Delta U)^2+4(\Delta S)^2}.$$

If we neglect squares of

$$\Delta T/T, \quad \Delta U/T, \quad \Delta S/T$$

it is easy to calculate that the retardation

$$2C_1T_1\alpha_1+2C_2T_2\alpha_2$$

has to be increased by

$$2C_1(\Delta T_1\alpha_1-\Delta U_1\alpha_1)+2C_2(\Delta T_2\alpha_2-\Delta U_2\alpha_2),$$

or, taking

$$C_1 = C_2 = C$$

in these corrective terms, the retardation must be increased by

$$2C[\Delta T_1\alpha_1-\Delta U_1\alpha_1+\Delta T_2\alpha_2-\Delta U_2\alpha_2],$$

and this is equivalent to putting in a *constant* correction W_0 to W.

PART II.

EXPERIMENTAL RESULTS.

§ 14. *Glasses Observed.*

The glasses used in this research were made for me by the firm of ZEISS in Jena. The makers being unable to communicate to me the chemical composition

of the glasses, the latter were analysed for me by Mr. W. J. REES, on the staff of Messrs. CHANCE BROS. To Mr. REES' skill I am indebted for the following results :—

Number of glass . .	1809.	3453.	2783.	3296.	935.	3413.	3749.
SiO_2	per cent. 35·4	per cent. 68·1	per cent. 52·7	per cent. 67·5	per cent. 32·5	per cent. 31·6	per cent. 70·2
PbO	18·7	—	31·6	—	28·2	23·6	—
Al_2O_3	3·7	—	0·6	—	8·5	8·0	—
ZnO	—	—	1·2	—	—	—	—
MgO	0·5	5·4	—	0·4	—	—	—
B_2O_3	34·3	5·7	1·4	15·4	27·7	33·0	5·9
K_2O	7·4	20·8	12·5	16·7	3·1	3·8	23·9

The majority of these glasses belong to the borosilicate variety, excepting 2783. 2783 is a flint glass, and was stated by the makers to be identical in composition with another glass, O 154, the composition of which (see 'Camb. Phil. Soc. Proc.,' vol. XII., Part V., p. 314) was stated by Messrs. ZEISS to include Na_2O and BaO. It seems probable that the composition of the later glass is a little different to that of O 154.

§ 15. *Linear Law connecting* λ *and the Stress.*

Since it was known beforehand that corrections to W of the type W_0+KW^2 would have to be applied, W_0 being due to the imperfect annealing, and KW^2 to relative and absolute rise and fall (see §§ 6, 13), instead of calculating the stress-optical coefficient C directly, as was done in previous experiments, the relation between W and λ was first studied, with a view to disengaging the corrections.

In practice, readings for W and λ were taken for both first and second orders of the band, and even, where possible, for third orders—for both tension and pressure. The tension and pressure observations were obtained by altering the relative heights of the two slabs by interchanging two steel slips which raised the supports of one of the slabs. The bending moments were not altered.

A typical set of results is embodied in Table I. below.

TABLE I.—Observations of Glass 1809.

A.				B.				C.			
W_1.	$\lambda_{obs.}$	$\Delta\lambda_{obs.}$	$3\Delta\lambda_{obs.}$	W_2.	$\lambda_{obs.}$	$\Delta\lambda_{obs.}$	$3\Delta\lambda_{obs.}$	W_3.	$\lambda_{obs.}$	$\Delta\lambda_{obs.}$	$2\Delta\lambda_{obs.}$
14·25	4506			26·25	4460			38·25	4480		
15·25	4891	385		28·25	4832	372		40·25	4705	225	
16·25	5206	315		30·25	5167	335		42·25	4909	204	429
17·25	5570	364	1064	32·25	5494	327	1034	44·25	5151	242	446
18·25	5895	325	1004	34·25	5820	326	988	46·25	5375	224	466
19·25	6265	370	1059	36·25	6180	360	1013				
20·25	6600	335	1030	38·25	6490	310	996				

A′.				B′.				C′.			
W_{-1}.	$\lambda_{obs.}$	$\Delta\lambda_{obs.}$	$3\Delta\lambda_{obs.}$	W_{-2}.	$\lambda_{obs.}$	$\Delta\lambda_{obs.}$	$3\Delta\lambda_{obs.}$	W_{-3}.	$\lambda_{obs.}$	$\Delta\lambda_{obs.}$	$3\Delta\lambda_{obs.}$
11·25	4430			24·25	4460			38·25	4570		
12·25	4750	320		26·25	4773	313		40·25	4815	245	
13·25	5000	250		28·25	5040	267		42·25	4975	160	
14·25	5310	310	880	30·25	5330	290	870	44·25	5172	197	602
15·25	5600	290	850	32·25	5618	288	845	46·25	5360	188	545
16·25	5890	290	890	34·25	5905	287	865	48·25	5560	200	585
17·25	6160	270	850	36·25	6215	310	885	50·25	5750	190	578
18·25	6460	300	860	38·25	6500	285	882				

The parts A, B, C refer to observations for tension: A′, B′, C′ to observations for pressure. In the columns headed W_n ($n = \pm 1, 2, 3$) are placed the observed weights, the suffix n indicating the order of the band observed, bands in pressure observations being taken of a *negative* order. The same notation will be kept throughout. The columns headed $\lambda_{obs.}$ contain the observed value of the wave-length of the light quenched, in tenth-metres. They are deduced from circle readings of the spectroscope, the law connecting these circle readings with the wave-lengths being obtained from observations of a known comparison spectrum. The spectrum of the arc between carbons soaked in calcium salt was used for this purpose.

Now, if we look at Table I., A., under the heading $\Delta\lambda_{obs.}$, we see that the differences of the observed λ for unit differences of W have a fairly constant average value, as is well shown on taking differences corresponding to differences of W of three units. This is done in the column headed $3\Delta\lambda_{obs.}$ of Table I., A.

It would seem, therefore, that the relation between λ and W is approximately linear. This impression is found to be confirmed when differences are taken in Table I., B, C, A′, B′, C′. In each case the differences are sensibly constant, especially if we bear in mind that an experimental error of 10 tenth-metres is to be expected.

There are some local inequalities, some of which will be shown later to be probably significant, but as a first approximation it seems we may assume a linear relation between λ and W.

Fig. 6 shows the observed λ plotted to W for the set of observations of Table I., B′. The observations obviously lie very close to the straight line given by the equation

$$\lambda = 949 + 145\mathrm{W}.$$

This equation was obtained by assuming a formula

$$\lambda = \lambda_0 + k\mathrm{W} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13).$$

$k = \dfrac{d\lambda}{d\mathrm{W}}$ was obtained by taking the mean value of the differences in the columns headed $3\Delta\lambda_{obs.}$ of Table I., B′, and λ_0 was then determined from the condition that the best straight line must pass through the centre of gravity of the observations.

The equation (13) clearly leads to the relation

$$\lambda(1 - \lambda_0/\lambda) = k\mathrm{W},$$

or, R being the relative retardation for a band of n^{th} order,

$$\mathrm{R} = nk\mathrm{W}/(1-\lambda_0/\lambda) = \mathrm{C}_0\mathrm{T}\tau/(1-\lambda_0/\lambda),$$

where T = effective tension, τ = thickness, C_0 = a constant independent of the wave-length.

Thus, the stress-optical coefficient C ($= \mu_1 - \mu_2$ for unit stress) is given by the approximate formula

$$C = C_0/(1-\lambda_0/\lambda) \quad \ldots \ldots \ldots \quad (14),$$

or

$$(C-C_0)(\lambda-\lambda_0) = C_0\lambda_0 \quad \ldots \ldots \ldots \quad (15).$$

The curve connecting C and λ is therefore a *rectangular hyperbola*. When $\lambda = \lambda_0$, $C = \infty$, and as λ increases without limit, C decreases to its limiting value C_0.

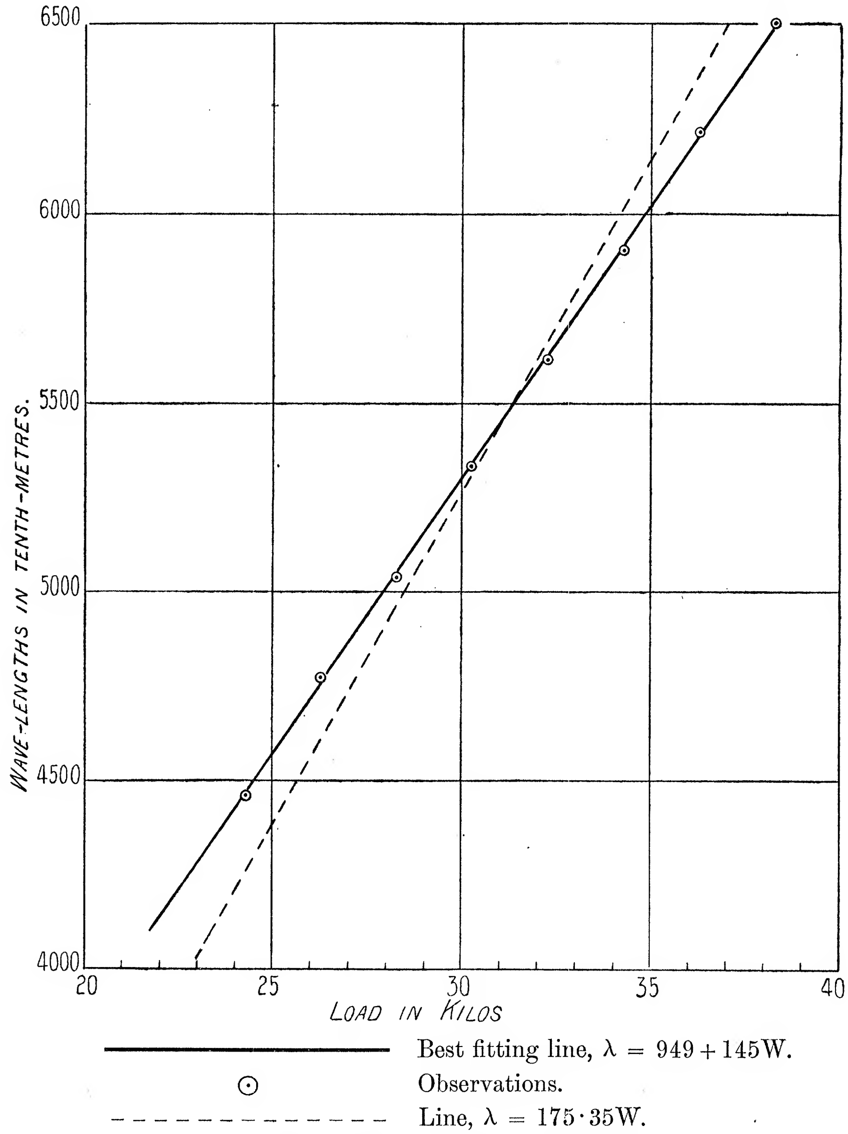

Fig. 6. Typical diagram showing relation of λ to W.

Thus, if the law continued to hold accurately for small wave-lengths, then for light of the critical wave-length λ_0 the stress-optical effect would become actually infinite.

No doubt this law is purely empirical so far, and will very probably not hold for very small wave-lengths. It is, however, sufficient to indicate that, as we approach

critical values in the ultra-violet, the stress-optical effect will very likely be largely increased.

In order to show the accuracy with which the observations above determine the value of λ_0, the straight line passing through the origin and through the centre of gravity of the observations has also been plotted on fig. 6.

Its equation is

$$\lambda = 175{\cdot}35 \,.\, W,$$

and on looking at the diagram it is obvious that no such straight line can fit the observations.

Further, it will be shown that all the observations, not merely of the glass 1809, but of the six other glasses examined, conform to a first approximation to the linear law.

§ 16. *Significance of this Linear Law.*

We have now to enquire how far this linear law has a physical meaning otherwise than as the expression of the trivial result that within a certain range of values *all* continuous variation is approximately represented by a straight line.

In previous papers, when C and λ were the quantities plotted, the relation was not well expressed by a straight line, the observations lying, in some cases, on a very decided curve (see 'Camb. Phil. Soc. Proc.,' vol. XII., Part V.). The observations which led to such curves were therefore re-reduced. The glasses selected which showed the effect most strongly were the Jena lead glasses O 152 and S 57.

Both of these were very closely fitted by a linear relation between λ and W. How close the fit was may be inferred from Table II. below, which shows the observed and calculated values of λ for one set of experiments with S 57, which is a very heavy flint glass, containing 80 per cent. of PbO.

In the table, W denotes pressure in kilogrammes applied by means of a compressing apparatus described in the paper referred to, and the entries in the column headed $\lambda_{\text{cal.}}$ are computed from the formula

$$\lambda = 3124{\cdot}9 + 4{\cdot}5136W.$$

The other sets of observations of S 57 and O 152, which have been re-reduced, show equally good agreement between the observed values of λ and those calculated from a formula of type (13).

Now the mean residual in Table II. is less than 5 tenth-metres, whereas the probable error of determination of the centre of a band is about 10 tenth-metres. Thus the law appears to fit the observations as closely as is possible within the limits of experimental error. It is worth noting that with these glasses, which contain a high percentage of lead, no deviations from the law, such as will be shown later to take place in some borosilicates, appear to exist.

Nevertheless one important experimental fact throws doubt on the universal validity of the linear law, even for lead glasses. POCKELS has shown ('Ann. d.

TABLE II.—Observations of S 57 (re-reduced).

W.	λ_{obs}	$\lambda_{cal.}$	$\lambda_{obs.} - \lambda_{cal.}$	$\lambda_{obs.} - \lambda_{hyp.}$	W.	$\lambda_{obs.}$	$\lambda_{cal.}$	$\lambda_{obs.} - \lambda_{cal.}$	$\lambda_{obs.} - \lambda_{hyp.}$
318·4	4550	4562	−12	−49	547·8	5592	5597	− 5	+ 9
343·4	4677	4675	+ 2	−23	572·9	5701	5711	−10	+ 6
364·0	4771	4768	+ 3	−13	595·0	5819	5810	+ 9	+23
389·2	4878	4882	− 4	−12	620·2	5923	5924	− 1	+ 9
409·8	4985	4975	+10	+10	639·5	6016	6011	+ 5	+12
435·0	5090	5088	+ 2	+ 7	664·7	6124	6125	− 1	+ 2
455·7	5178	5182	− 4	+ 6	685·7	6216	6220	− 4	− 8
480·8	5300	5295	+ 5	+20	710·8	6341	6333	+ 8	− 3
501·5	5387	5388	− 1	+13	735·6	6447	6445	+ 2	−19
526·6	5500	5502	− 2	+15	—	—	—	—	—

Physik,' 1902, p. 745) that for a glass containing between 60 and 70 per cent. of PbO the stress-optical coefficient changes sign, and an experiment made by him with such a glass pointed to the fact that the stress-optical coefficient did not vanish simultaneously for all colours, a result which has been independently confirmed by the present author from considerations of curves showing C and $\Delta C/\Delta\lambda$ plotted to percentage of lead (see 'Camb. Phil. Soc. Proc.,' vol. XXI., p. 335). Now if the law $C = C_0/(1-\lambda_0/\lambda)$ held universally, the vanishing of C_0 would imply the vanishing of C for every wave-length.

Moreover, it seems impossible to find theoretical justification for such a formula. It is well known (see DRUDE'S 'Theory of Optics,' cap. V., pp. 388, 389)* that the index of refraction μ is given by the formula $\mu^2 = 1+\Sigma A_p/\{1-(\lambda_p/\lambda)^2\}$, where λ_p = wave-length *in vacuo* of light belonging to one of the natural periods of the glass.

[**Note added July 3rd*, 1907.—Throughout the paper I have followed DRUDE. But if we adopt LORENTZ'S formula, viz. :—

$$(\mu^2-1)/(\mu^2+2) = \Sigma A_p/\{1-(\lambda_p/\lambda)^2\},$$

or similar formulæ, the most essential part of the reasoning remains in most cases practically unaltered.]

It seems, at first sight, highly probable that the effect of stress will be, not to introduce different free periods of the atoms for differently polarized rays—that is, not to alter λ_p—but to change the coefficient A_p, which depends on the number and arrangement of the molecules.

This will lead to the result

$$2\mu\, \delta\mu = \Sigma\, \delta A_p/\{1-(\lambda_p/\lambda)^2\}$$

or

$$\mu C = \Sigma C_p/\{1-(\lambda_p/\lambda)^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

Now μ itself, when expanded in powers of $1/\lambda$, will involve terms in λ^{-2}, λ^{-4}, etc. Therefore C will involve only such even terms. Hence no formula involving λ to odd powers can be theoretically acceptable.

If we suppose that one term, corresponding to wave-length λ_p, is active in producing the dispersion, both in ordinary refraction and artificial double-refraction, we have

$$\mu C = C_p/\{1-(\lambda_p/\lambda)^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

$$\mu^2 = \mu_0{}^2 + A_p\lambda_p{}^2/(\lambda^2-\lambda_p{}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

The formula (17) is open to the same objection as $C = C_0/(1-\lambda_0/\lambda)$, namely, that it does not satisfy the case of a glass where the double-refraction vanishes for one wave-length without the dispersion vanishing at the same time. It is clear that in this case other free periods, whose effect is usually negligible, become important.

For other glasses, however, the formulæ (17) and (18) might be good approximations. To get μ from (18) remember that for wave-lengths greater than 4300 the dispersion terms are $< \frac{1}{10}$ of the whole. Then, using the Binomial Theorem, we find that, to an accuracy of $\frac{1}{1000}$ nearly,

$$\begin{aligned} \mu &= \mu_0 + A_p\lambda_p{}^2/2\mu_0\,(\lambda^2-\lambda_p{}^2) \\ &= \mu'_0 + A'_p/\{1-(\lambda_p/\lambda)^2\}. \end{aligned}$$

Hence

$$C = C_p/[A'_p + \mu'_0\{1-(\lambda_p/\lambda)^2\}] = C'_0/\{1-(\lambda'_p/\lambda)^2\} \quad . \quad . \quad . \quad . \quad (19),$$

where

$$\left.\begin{aligned} C'_0 &= C_p/(\mu'_0 + A'_p) \\ \lambda'_p &= \lambda_p\sqrt{\mu'_0/(\mu'_0+A'_p)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

A formula of the type (19) for C would lead to a curve connecting the wave-length of extinction and the load of the type

$$(\lambda - kW)\,\lambda = (\lambda'_p)^2.$$

In general, when λ'_p and λ_0 are small, it will be found that either formula,

$$C = C_0/\{1-(\lambda_0/\lambda)\}, \qquad C = C'_0/\{1-(\lambda'_p/\lambda)^2\},$$

represents the observations almost equally well.

But in the case of the results of Table II. a hyperbola was fitted to the observations, its equation being

$$\lambda-(3460)^2/\lambda = (6\cdot2489)\,\mathrm{W}.$$

The differences between λ as calculated from the above formula and λ observed are given in Table II. under the heading $\lambda_{\text{obs.}}-\lambda_{\text{hyp.}}$ The mean value of the residuals taken without regard to sign is between 13 and 14, or nearly three times the value of the mean residual from the straight line.

Even this mean residual hardly exceeds the probable error of observation, so that this would not be conclusive against the hyperbola. But an examination of the individual numbers in the last column of Table II. shows strong systematic positive residuals in the middle and negative residuals at the ends, and these systematic divergences certainly suggest that the hyperbola is not the most suitable curve.

The index of refraction of this particular glass is tolerably represented for the visible spectrum by the formula

$$\mu = 1\cdot5107+0\cdot39125/\{1-(2159\cdot6/\lambda)^2\}.$$

Thus

$$\lambda_p = 2159\cdot6, \quad \mu'_0 = 1\cdot5107, \quad \mathrm{A}'_p = 0\cdot39125.$$

From this λ'_p of formula (20) comes out to be $1924\cdot7$. This differs entirely from the value obtained from the experiments, namely 3460. We are thus led to the interesting conclusion that in this glass at least the free periods which produce the ordinary dispersion are probably not active in producing the dispersion of artificial double refraction.

This removes theoretical justification in this case for the formula

$$\mathrm{C} = \mathrm{C}'_0/\{1-(\lambda'_p/\lambda)^2\},$$

even if it had not been shown inferior as a purely empirical fit.

We may then provisionally accept the law

$$\mathrm{C} = \mathrm{C}_0/(1-\lambda_0/\lambda),$$

and the results in what follows will be reduced with reference to it.

At the same time it must be remembered that the physically significant formula is probably of type (16). It will be shown in § 21 that even in the visible spectrum there are local divergences from the linear law.

§ 17. *Methods of Reduction.*

In fitting a linear law

$$\lambda = \lambda_0+k'\mathrm{W}$$

to a set of observations, the corrections due to the sinking and permanent stress had to be taken into account.

The correct value for the load is given by

$$W_0 + W + \gamma W^2.$$

It is therefore a formula

$$\lambda = \lambda_0 + k'(W_0 + W + \gamma W^2)$$

which has to be fitted.

Assuming for the present that for monochromatic light the relative retardation is strictly proportional to the load, we have, for the band of the r^{th} order, $k' = k/r$, where k is the same constant for observations of all orders. Thus

$$r\lambda = r\lambda_0 + kW_0 + kW + k\gamma W^2.$$

Suppose we have p observations, the first step is to take a number of differences $\Delta\lambda$ corresponding to differences ΔW sufficiently large to minimise the effects of accidental irregularities and to form the fraction

$$\Sigma(\Delta\lambda)/\Sigma(\Delta W) = \{(\Delta\lambda)/(\Delta W)\}_r,$$

the suffix r denoting that the band observed is of the r^{th} order.

If our p observations correspond to values of W differing by a constant increment ΔW, and if we take differences of λ corresponding to differences $q\Delta W$, we obtain $p-q$ equations—

$$r(\lambda_{q+1} - \lambda_1) = kq\Delta W + k\gamma q\Delta W(2W + q\Delta W),$$

$$r(\lambda_{q+2} - \lambda_2) = kq\Delta W + k\gamma q\Delta W(2W + \overline{q+2}\Delta W),$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$r(\lambda_p - \lambda_{p-q}) = kq\Delta W + k\gamma q\Delta W(2W + \overline{2p-q-2}\Delta W).$$

If these be added up, we have

$$r\sum_{s=1}^{s=p-q}(\lambda_{q+s} - \lambda_s) = kq(p-q)\Delta W + k\gamma q\Delta W(p-q)(2W + \overline{p-1}\Delta W),$$

$$r\left[\frac{1}{q(p-q)}\sum_{s=1}^{s=p-q}(\lambda_{q+s} - \lambda_s)\right] \div \Delta W = k + 2k\gamma\overline{W},$$

where $\overline{W}$ = mean value of W.

We have then

$$r(\Delta\lambda/\Delta W)_r = k + 2k\gamma\overline{W} \quad \ldots\ldots\ldots \quad (21).$$

A comparison of the values of $r(\Delta\lambda/\Delta W)_r$ then enables us at once to discover whether a correction γW^2 is needed for the observations or not.

For most of the glasses examined the values of $r(\Delta\lambda/\Delta W)_r$ do not indicate a correction of this type of sensible amount. In doing the reductions for such glasses γ has been taken equal to zero.

For one glass γ has a sensible value. In this case a suitable value of γ having been

formed, corrections were applied to the observed load and the observations re-reduced with $\gamma = 0$.

From this point onward, therefore, γ may be taken zero in the reductions. k is then equal to $r(\Delta\lambda/\Delta W)_r$.

In practice the values of $r(\Delta\lambda/\Delta W)_r$ vary slightly with different r's. In most cases, however, a sufficiently good fit is obtained by taking for k the mean value of $r(\Delta\lambda/\Delta W)_r$ and reducing the observations of different orders by means of this single value. In one glass HOOKE's law did not seem to hold quite exactly, and the observations of different orders were reduced independently.

k having been determined, $\lambda_0 + \frac{k}{r} W_0$ is found from the condition that the best fitting straight line

$$\lambda = \lambda_0 + \frac{k}{r} W_0 + \frac{k}{r} W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

must be satisfied by the mean values $\lambda = \bar{\lambda}$, $W = \overline{W}$.

We thus obtain equations

$$\Lambda_1 = \lambda_0 + kW_0 = \bar{\lambda}_1 - k\overline{W}_1,$$

$$2\Lambda_2 = 2\lambda_0 + kW_0 = 2\bar{\lambda}_2 - k\overline{W}_2,$$

$$3\Lambda_3 = 3\lambda_0 + kW_0 = 3\bar{\lambda}_3 - k\overline{W}_3,$$

etc.

Two of these equations are theoretically sufficient to determine λ_0 and W_0. In practice three are often obtained. The three equations are then solved by least squares. The solution is given by

$$\lambda_0 = (3\Lambda_3 - \Lambda_1)/2,$$

$$kW_0 = (\Lambda_1 + 2\Lambda_2 + 3\Lambda_3)/3 - 2\lambda_0.$$

From the values of k, λ_0, W_0 so determined λ has been computed from the formula (22) and compared with the observed value.

§ 18. *Tables of Results.*

The following gives a table of the constants λ_0, kW_0, k for the various sets of observations. Observations corresponding to tension and pressure are distinguished by the letters T, P respectively.

When λ_0, k, kW_0 are known, the wave-length of the band of r^{th} order is computed from the load by the formula (22). The average discrepancy in tenth-metres between λ thus calculated and λ observed, for each set of observations, is entered in the column headed (O−C).

TABLE III.

Glass.	λ_0.	kW_0.	k.	O – C.
1809 T	352	– 670	340·25	15
1809 P	770	407	389·07	11
3296 T	436	83	323·89	14
3296 P	638	– 76	316·58	20
3453 T	439	113	249·61	18
3453 P	609	– 89	246·52	17
3413 T	419	20	311·68	14
3413 P	687	– 69	299·24	6
3749 T	405	194	258·48	12
3749 P	724	– 92	249·04	18
935 T	183	– 1320	376·19	27
935 P	719	929	252·21	30

For the glass 935 a correction γW^2 was applied to W, γ being taken $+0·001$ for pressure and $-0·001$ for tension. In (22) we have then to substitute $W+\gamma W^2$ for W. This glass is badly annealed and does not seem well fitted by the formula.

The glass 2783 had to be reduced differently. This is a lead glass, a specimen of which had been examined under simple pressure and whose behaviour had appeared peculiar (see 'Camb. Phil. Soc. Proc.,' vol. XII., Part V., p. 323, where the glass in question is described as O 154).

There are two sets of tension observations, denoted by A, B in Table IV., and two sets of pressure observations denoted by C, D. The values of Λ_r, $r\Delta\lambda/\Delta W_r$ (see § 17) are given in Table IV.

TABLE IV.

Set	r.	Λ_r.	$r\Delta\lambda/\Delta W_r$.	O – C.
A	1	386	256·45	18
	2	351	262·30	8
B	1	507	255·54	18
	2	443	262·80	11
C	1	958	236·70	25
	2	687	245·96	17
	3	608	252·00	16
D	1	768	243·58	14
	2	675	247·36	16

In this case there seems to be a progressive increase of $r\Delta\lambda/\Delta W_r$ with the order, for both tension and pressure. This excludes a correction for sinking, since the latter must act opposite ways for tension and pressure. It is here probably due to a failure of HOOKE's law, which the observations have shown otherwise, and which will be discussed in a later section.

Under the circumstances no real advantage could be derived here by attempting to reduce the various sets by means of a single formula. The sets have, therefore, been independently reduced, using the formula

$$\lambda = \Lambda_r + W\,(\Delta\lambda/\Delta W_r).$$

The actual observations of all glasses are given in Table V. for purposes of reference. Each column corresponds to a single set of observations. As a rule the order of the band observed will be clear from the place of the observation in the series. Wherever this is not so, or where observations of different orders correspond to the same load, Roman numerals have been added to indicate the order of the band.

TABLE V.

W.	1809.		3296.		3453.		3413.		3749.				935.		2783.			
	T.	P.	T.	P.	T.	P.	T.	P.	T.	T.	P.	P.	T.	P.	T.	T.	P.	P.
11·25		4430												4549				
11·75			4321															
12·25		4750	4519	4465										4817				
13·25		5000	4826	4791			4610	4598						4963				
14·25	4506	5310	5083	5064			4868	4875						5230				
15·25	4891	5600	5449	5386	4312	4249	5182	5182		4558	4375	4380	4553	5517	4271	4357	4546	4507
16·25	5206	5890	5786	5706	4652	4543	5485	5466	4800	4812	4680	4705	4881	5759	4581	4715	4824	4750
17·25	5570	6160	6085	6027	4874	4793	5812	5789		5063	4900		5280	6036	4830	4924	5082	4938
18·25	5895	6460	6444	6379	5056	5000	6119	6079	5325	5312	5160	5200	5633	6375	5041	5143	5250	5213
19·25	6265				5343	5273	6418	6383		5557	5425		5987	6596	5323	5436	5494	5444
20·25	6600				5618	5508			5846	5840	5675	5710	6367		5572	5689	5759	5698
21·25					5855	5747			6085	6089	5900		6663		5858	5936	5963	5936
22·25					6105	5987			6340	6340	6165	6130			6106	6197	6231	6190
23·25			4232		6402	6241				6565	6450				6345	6452	6483	6425
24·25		4460	4401	4409	6613	6549						6680		4298	6569	6687	6638	6703
26·25	4460	4773	4746	4780			4524	4576					4274	4626				
28·25	4832	5040	5038	5045			4836	4881					4655	4874				
29·25									4278		4285	4300						
30·25	5167	5330	5378	5366	4270	4264	5111	5185	4377	4374	4495	4430	5024	5100	4295	4412	4411	4429
32·25	5494	5618	5701	5698	4547	4570	5449	5464	4685	4687	4680	4700	5376	5380	4600	4680	4687	4671
34·25	5820	5905	6020	6000	4804	4818	5786	5771	4924	4928	4930	4935	5717	5670	4841	4928	4904	4908
35·25			4240 III.															
36·25	6180	6215	6372 II.	6340 II.	5013	5020	6092	6071	5189	5201	5160	5185	6076	5942	5107	5220	5113	5157
36·25			4361 III.	4466 III.														
38·25	6490 II.	6500 II.	6650 II.	6630 II.	5270	5281	6418 II.	6391 II.	5424	5432	5425	5445	6410 II.	6208 II.	5372	5469	5374	5394
38·25	4480 III.	4570 III.					4379 III.	4480 III.					4321 III.	4327 III.				
39·25			4721 III.	4811 III.									4463 III.	4437 III.				
40·25					5508	5505		6660 II.	5706	5712	5680	5695	6696 II.	6514 II.	5632	5750	5635	5635
40·25	4705 III.	4815 III.					4615 III.	4676 III.										
42·25	4909	4975	5017	5059	5768	5768	4834	4879	5957	5996	5930	5945	4827	4756	5900	6003	5885	5888
44·25	5151	5172			5996	6006	5024	5077	6218	6241	6175	6220			6157	6267	6135 II.	6125
44·25																	4299 III.	
45·25			5344	5366									5164	4993				
46·25	5375	5360			6263	6274	5234	5281	6464	6483		6465			6418	6527	6410 II.	6387
46·25																	4501 III.	
48·25		5560	5671	5689	6507	6518	5430	5473					5485	5283	6663	6750	6588 II.	6703
48·25																	4680 III.	
50·25		5750					5635	5676					5715				4843	
51·25			6003	6010									5840	5537				
52·25																	4996	
54·25																	5153	

§ 19. *Discussion of the Values of* λ_0, $k\mathrm{W}_0$.

The first thing which strikes the eye on looking through the results of the last section is, that although tension observations of different orders and pressure observations of different orders are fairly well fitted by the same λ_0, W_0, and k, the same does not hold of tension and pressure observations taken together.

The differences in k are only what should have been expected, since k depends on the adjustments.

With regard to $k\mathrm{W}_0$ the values for pressure and tension should theoretically be equal and opposite. For if light traverse a thickness τ of glass in which a residual tension T_0 exists, a term $\mathrm{CT}_0\tau$ is added to the relative retardation when external tension is applied and subtracted from it when external pressure is applied.

Again λ_0 should be a constant for the glass, and therefore the same for tension and pressure, if the stress-optical coefficient be independent of the nature and magnitude of the load applied.

Now Table III. shows clearly that, although the values of $k\mathrm{W}_0$ differ in sign, they are only very roughly of the same order of magnitude.

Possibly this might be accounted for by the fact that in different experiments the light did not pass through the same parts of the glass, so that the value of the residual stress might have been different.

The divergences in λ_0 are considerable; λ_0 appears systematically larger for pressure than for tension.*

§ 20. *Systematic Residuals.*

The residuals $\lambda_{\text{obs.}} - \lambda_{\text{cal.}}$ have in all cases been plotted on a large scale against $\lambda_{\text{obs.}}$ Three of these diagrams are shown in figs. 7 to 9. The pressure and tension observations have been plotted to a different base in each case, to avoid the diagrams overlapping, so that two zero-marks appear on each scale of residuals. Of these the upper zero mark refers to tension observations.

Most of them (*e.g.*, fig. 9) are fairly irregular, which is not surprising when we bear in mind that an error of 1 division in the ordinate (10 tenth-metres) is the probable error of the observations.

Two glasses, however, 3296 and 3453, figs. 7 and 8, appear to show very strongly systematic residuals between 4200 and 5500. If we look at fig. 7 we notice that the curves rise from 4200 to a peak about 4700, after which there is a sharp fall with a trough about 5050.

After this the curves run fairly horizontal, with indications of another peak at 6300.

[* *Note added April 3rd*, 1907.—Later experiments do not confirm the systematic difference between the values of λ_0 for tension and pressure. Very probably the divergence previously noted was due to a change in the adjustments which had to be made when passing from tension to pressure, and which rendered the observations of the two kinds not strictly comparable.

The true value of λ_0 appears to be the mean of the values obtained for tension and pressure.]

The course of the diagrams in fig. 8 is very similar. There is a well marked peak about 4700, followed by a depression in the neighbourhood of 5000. There are also slight indications of a depression at 6000, with a subsequent rise.

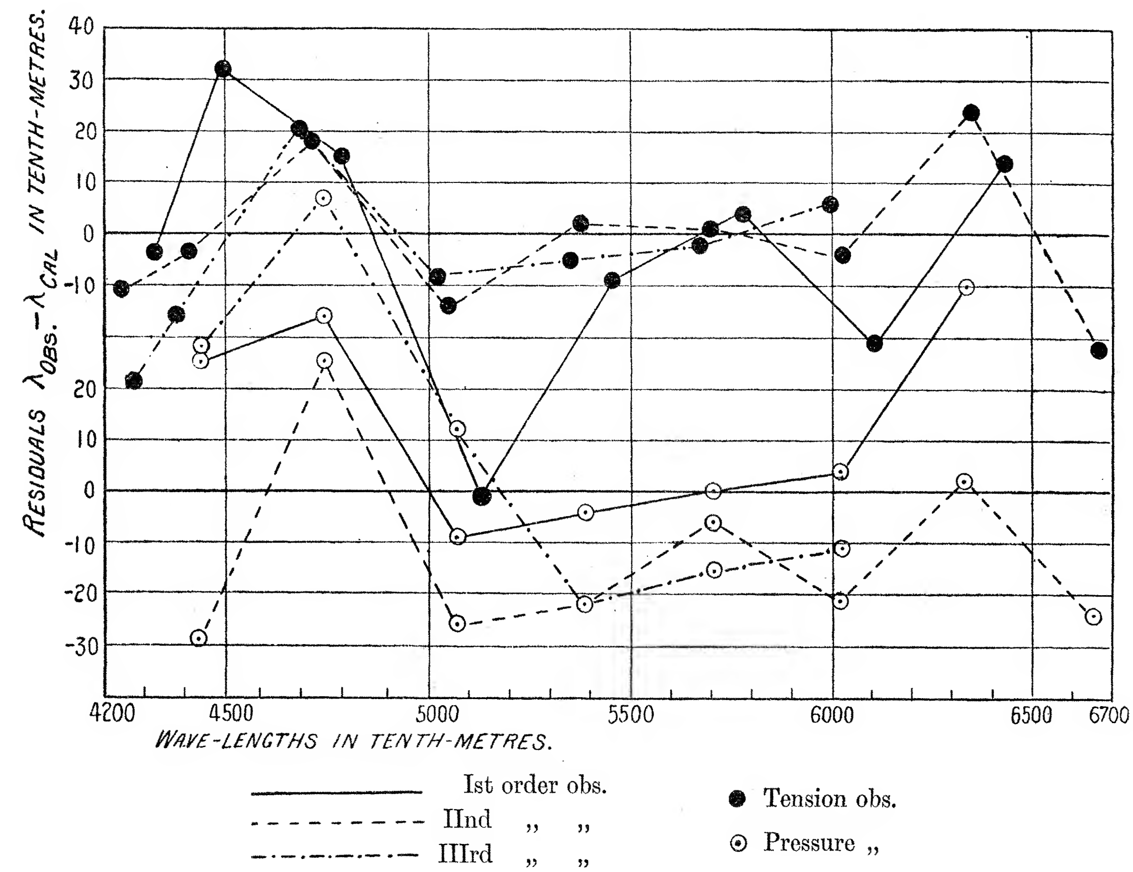

Fig. 7. Glasses 3296. Diagram of residuals.

These systematic residuals, which are in most cases quite large, and which are shown in the same place by all the tension and pressure observations of these glasses, cannot be chance effects. Neither can they be affected, denoting, as they do, comparatively rapid changes in λ, by any of the slowly varying corrections which have been discussed. They can be accounted for only in the following ways:—

(1) Possible erroneous identification of a spectrum reference line in the neighbourhood and consequent wrong determination of λ observed;
(2) Bad division errors of the spectroscope circle;
(3) Bad error in some of the weights employed at these points;
(4) Systematic change of personality of observer in this neighbourhood, due to change of colour;
(5) Actual variation of the law of stress-optical effect in this neighbourhood.

(1) is ruled out by the fact that in the glass 3453, where the effect was first

noticed, it was discovered, not from any such curve, but from the actual circle readings of the spectroscope, which usually increased by steadily increasing differences; in this glass, just after the readings corresponding to λ 5000, the difference decreased instead of increasing. Readings were taken several times with great care, and the effect was confirmed in each case. This demonstrated that the cause was not accidental.

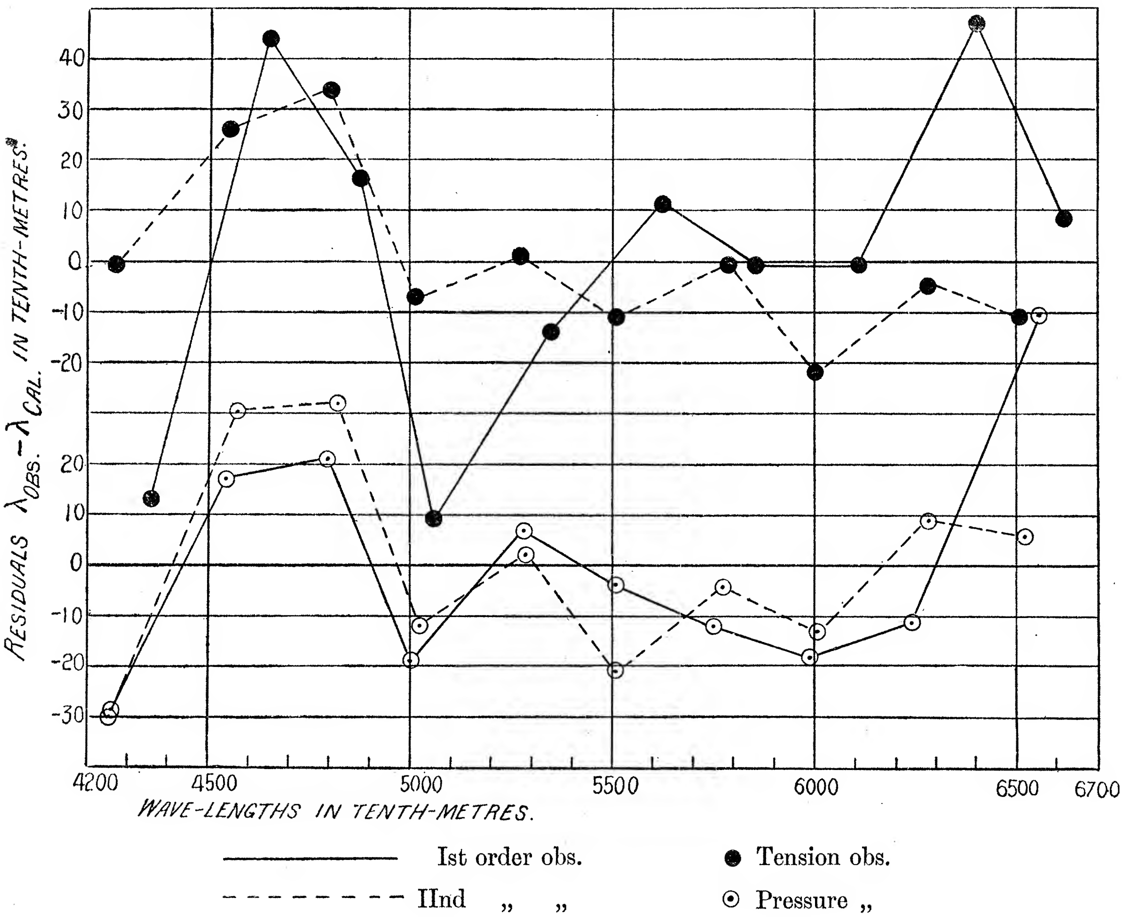

Fig. 8. Glasses 3453. Diagram of residuals.

Clearly a wrong determination of a reference line is out of the question; this could not cause an irregularity in the differences of circle reading.

With regard to (2) and (3), the weights and the divisions of the circle were tested with great care and found correct.

As to personality, the jump of 40 tenth-metres between λ 5000 and λ 4800 would require an error of 6′ in locating the centre of the band, and a change of personality of this amount, in a fairly bright region of the spectrum, is unthinkable. Besides, if the effect is due to such a change, it should appear in all the glasses, which is not the case.

Under these conditions it seems safe to assert that between wave-lengths 4500 and

5500 there exists a definite deviation from the straight line law, which deviation takes the shape of an undulation, with a crest at about 4700 and a trough between 5000 and 5100.

To trace this effect more exactly, a new set of readings were taken with the glasses 3453. The readings were taken with special care at intervals of load of half a kilogramme. The residuals from the best straight line were computed as before, and they are

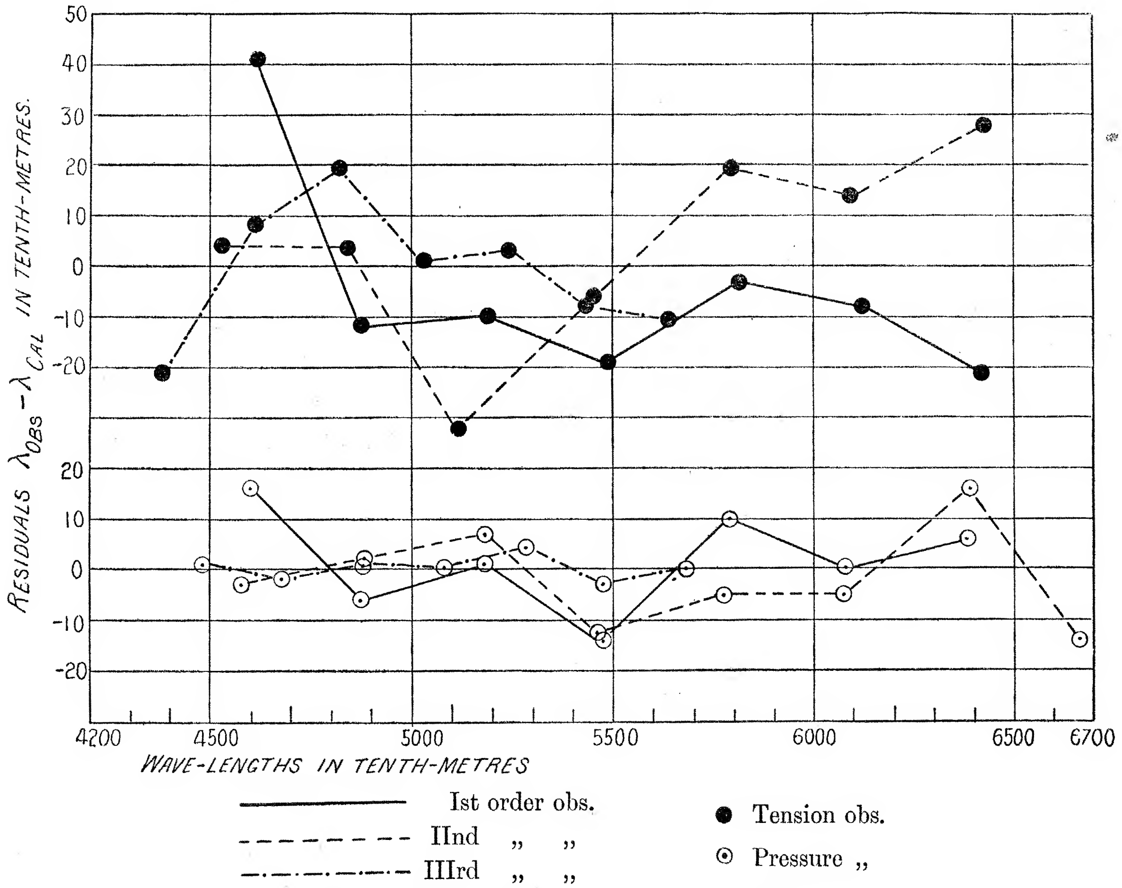

Fig. 9. Glasses 3413. Diagram of residuals.

shown on fig. 10. Here, the observations being much more numerous, the residuals indicate a distinct curve. The dotted curve in the figure has been drawn freehand through the points to give some idea of the general shape of this curve.

It is seen that this curve amply confirms the previous set of results, although the observations were taken at several months' interval, with different adjustments, and probably different personal equations.

On examination, only 3296 shows anything like so marked an effect. 1809 and 2783 show the effect in much the same place, but weakly, and some sets of observations do not confirm it. 935 and 3749 are hopelessly irregular. Nothing definite can be asserted about them. As to 3413, the pressure observations do not show this effect at all, and the tension observations show it only very doubtfully.

There are indications, however, in this glass, of a systematic dip at 5500, and a subsequent rise. Various glasses also show signs of a peak in the red, between 6200 and 6500. None of these, however, are more than mere indications, and it is only the curves for 3453, and in particular fig. 10, on which any safe deductions and measurements can be based.

If we refer to the table of § 14 we see that 3453 and 3296 are very much alike in chemical composition. Apart from this, no relation between this effect and chemical composition can be predicated. It seems almost certain that boric acid has nothing to do with it. The glasses richest in B_2O_3 do not show the effect. K_2O can hardly be the explanation, or 3749 should show the effect more strongly. It seems not at all unlikely that a small impurity, such as magnesium or zinc oxide, may be the cause of the result. It is noteworthy that the only glasses which seem to show the effect at all definitely, are precisely those which contain MgO and ZnO, and that the one which really shows the effect in a measurable manner contains quite a respectable percentage of MgO.

§ 21. *Possible Explanation by Absorption Bands.*

The shape of the curve of residuals resembles the curve of index of refraction plotted to either period or wave-length when we pass through an absorption band. This suggests that the effect may be due to some faint or latent absorption band of the glass in the visible spectrum, which band corresponds to a period active in producing the artificial double-refraction.

Following, as before, DRUDE ('Theory of Optics,' cap. V.), we have, μ being the index of refraction, and κ the co-efficient of absorption,

$$\mu^2 (1-i\kappa)^2 = \text{terms not depending upon the absorption band}$$
$$+Q/\{1+i\alpha/\lambda-(\lambda_p/\lambda)^2\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23),$$

where λ_p is the wave-length of the absorption band, α is a coefficient which increases with the absorption, and Q is a coefficient depending on the arrangement and number of the electrons.

This leads to

$$\mu^2 (1-\kappa^2) = \mu_0{}^2+Q\,\{1-(\lambda_p/\lambda)^2\}/[\{1-(\lambda_p/\lambda)^2\}^2+\alpha^2/\lambda^2]$$
$$2\mu^2\kappa = (Q\alpha/\lambda)/[\{1-(\lambda_p/\lambda)^2\}^2+\alpha^2/\lambda^2].$$

Now suppose the stress T to leave λ_p and α unaltered, and to alter Q.

$$2\mu (1-\kappa^2)\, d\mu/dT-2\mu^2\kappa\, d\kappa/dT = 2\mu_0\, d\mu_0/dT+\{1-(\lambda_p/\lambda)^2\}(dQ/dT)/[\{1-(\lambda_p/\lambda)^2\}^2+\alpha^2/\lambda^2],$$
$$4\mu\kappa\, d\mu/dT+2\mu^2 d\kappa/dT = (\alpha/\lambda)\,(dQ/dT)/[\{1-(\lambda_p/\lambda)^2\}^2+\alpha^2/\lambda^2],$$

whence, eliminating $d\kappa/dT$,

$$2\mu (1+\kappa^2)\, d\mu/dT = 2\mu_0 d\mu_0/dT+\{1-(\lambda_p/\lambda)^2+\kappa\alpha/\lambda\}(dQ/dT)/[\{1-(\lambda_p/\lambda)^2\}^2+\alpha^2/\lambda^2] \quad . \quad (24).$$

To simplify the calculation, we may suppose that in the initial state of things $Q = 0$, so that $\kappa = 0$, $\mu = \mu_0$. As a matter of fact Q cannot be 0, but the assumption that $\kappa = 0$, $\mu = \mu_0$ is good enough to give the characteristics of the phenomenon sufficiently well for our purpose.

We have then

$$d\mu/d\mathrm{T} = d\mu_0/d\mathrm{T} + (d\mathrm{Q}/d\mathrm{T})\{1-(\lambda_p/\lambda)^2\}/2\mu_0[\{1-(\lambda_p/\lambda)^2\}^2+\alpha^2/\lambda^2].$$

In all the above $d/d\mathrm{T}$ denotes rate of divergence with regard to T of a quantity for the two oppositely polarized rays. Thus

$$d\mu/d\mathrm{T} = \mathrm{C},$$

$$d\mu_0/d\mathrm{T} = \text{value of C if there were no absorption band.}$$

Accordingly,

$$\tfrac{1}{2}(d\mathrm{Q}/d\mathrm{T})\{1-(\lambda_p/\lambda)^2\}\mu_0^{-1}\{[1-(\lambda_p/\lambda)^2]^2+\alpha^2/\lambda^2\}^{-1}$$

gives the deviation produced in the stress-optical coefficient by the absorption band. Calling this $\delta\mathrm{C}$, we have

$$\lambda_{\text{obs.}} : \lambda_{\text{cal.}} = \mathrm{C}+\delta\mathrm{C} : \mathrm{C}.$$

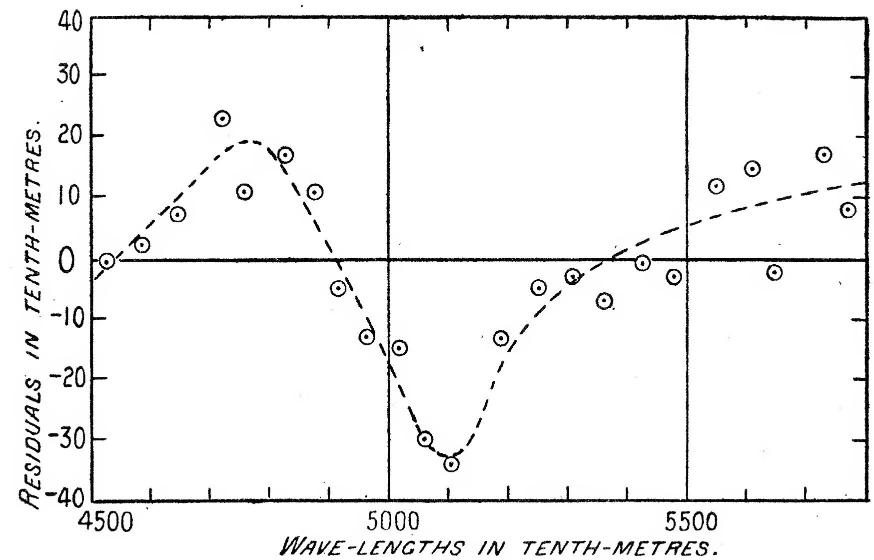

Fig. 10. Diagram showing curve of residuals from straight line for glass 3453.

Thus the deviation $\lambda_{\text{obs.}}-\lambda_{\text{cal.}}$ which is given by fig. 10 is $\lambda\delta\mathrm{C}/\mathrm{C}$.

λ, μ_0, C are all comparatively slowly varying: the factor which causes the oscillation is

$$\{1-(\lambda_p/\lambda)^2\}\{[1-(\lambda_p/\lambda)^2]^2+\alpha^2/\lambda^2\}^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

This factor starts with the value 0 when $\lambda = 0$, decreases to a negative minimum $-\lambda_p^2/\alpha(2\lambda_p+\alpha)$ when $\lambda^2 = \lambda_p^3/(\lambda_p+\alpha)$, and then increases to a positive maximum $\lambda^2_p/\alpha(2\lambda_p-\alpha)$ when $\lambda^2 = \lambda^3_p/(\lambda_p-\alpha)$. It then decreases down to 1 when $\lambda = \infty$.

α being here small with regard to λ_p, the wave-lengths of minimum and maximum are approximately $\lambda_p - \frac{1}{2}\alpha$, $\lambda_p + \frac{1}{2}\alpha$. Neglecting the variations of the other factors, this result gives us an easy means of obtaining α from fig. 10. α is the horizontal distance between the maximum and the following minimum. α is therefore between 300 and 400 tenth-metres—say 350 ; λ_p from the same diagram is about 4900.

This phenomenon also gives us experimental evidence in favour of a non-alteration of the period. If we refer to the physical meaning of Q, we find it to be

$$Q = Ne^2\tau^2/m\pi,$$

where

τ = period of the light corresponding to the absorption band,

N = number of electrons in unit volume which vibrate in this particular mode,

e = charge on such an electron,

m = its mass.

It follows that when we suppose τ, that is λ_p, to vary (κ being initially zero, as before), what we have called δC is given by

$$\begin{aligned} &\tfrac{1}{2}\mu_0^{-1}(d/dT)\,\{Q\,[1-(\lambda_p/\lambda)^2]/([1-(\lambda_p/\lambda)^2]^2+\alpha^2/\lambda^2)\} \\ &\quad = (Ne^2/2\mu_0 mv^2\pi)\,(d\lambda_p^2/dT)\,d/d\,(\lambda_p^2)\left\{\frac{\lambda_p^2\lambda^2(\lambda^2-\lambda_p^2)}{(\lambda^2-\lambda_p^2)^2+\alpha^2\lambda^2}\right\} \\ &\quad = (Ne^2/2\mu_0 mv^2\pi)\,(d\lambda_p^2/dT)\left\{\frac{\lambda^4\,[(\lambda^2-\lambda_p^2)^2-\alpha^2(2\lambda_p^2-\lambda^2)]}{[(\lambda^2-\lambda_p^2)^2+\alpha^2\lambda^2]^2}\right\} \quad . \quad . \quad . \quad (26). \end{aligned}$$

Now, in the neighbourhood of $\lambda = \lambda_p$, α being small, the rapidly varying terms which determine the shape of the curve are those involving $\lambda^2 - \lambda_p^2$. If we call this quantity x, and put $\lambda = \lambda_p$ in the other terms, we obtain some conception of the shape of the curve. Taking

$$y = (x^2-\alpha^2\lambda_p^2)\,(x^2+\alpha^2\lambda_p^2)^{-2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27),$$

y has one minimum value $-(\alpha^2\lambda_p^2)^{-1}$ when $x = 0$. It has two maximum values $(\alpha^2\lambda_p^2)^{-1}/8$ when $x = \pm\sqrt{3}\alpha\lambda_p$ or $\lambda = \lambda_p \pm \sqrt{3}\alpha/2$. The general shape of the curve is shown in fig. 11. It is at once obvious that it does not show the alternate large maximum and minimum required to fit the curve of fig. 10. So far, then, the experiments bear out the hypothesis that the free periods are not altered.

Before we can proceed further we have to settle finally the convention about the sign of C. It has been usual to call C positive for ordinary glass, such as that investigated by BREWSTER and KERR, and C negative for heavy flint like S 57. This convention has been adopted by the author in previous papers.

We will now rigidly define the stress-optical coefficient as

$$C = |(\mu_{T_2} - \mu_{T_1})/(T_2 - T_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28),$$

where μ_T denotes the index of refraction of the ray *vibrating* along the direction in which the principal stress (tension being considered positive, as usual) is T.

Since the direction of vibration—if we take the electric force to be the light-vector —is perpendicular to the direction of polarization, and the index of refraction is inversely proportional to the velocity, this means that a positive C implies a higher

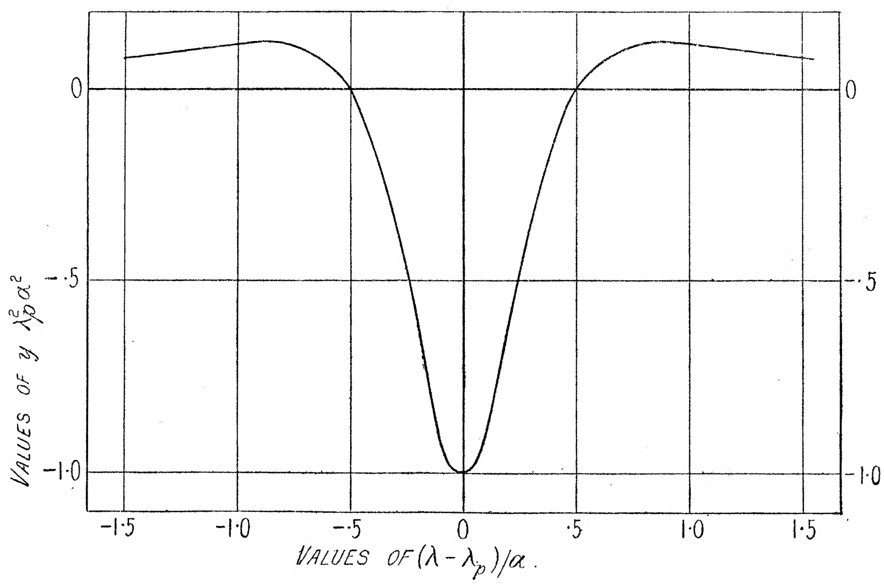

Fig. 11. Diagram of the curve $y = [(\lambda^2 - \lambda_p{}^2)^2 - \alpha^2\lambda_p{}^2]\,[(\lambda^2 - \lambda_p{}^2)^2 + \alpha^2\lambda_p{}^2]^{-2}$.

velocity in the ray polarized in the direction of greatest stress. Now this ray corresponds to the ordinary ray if the glass be compared with a uniaxial crystal whose optic axis is parallel to the direction of greatest stress.

The glass, therefore, produces the same effect as a *positive* uniaxial crystal so placed.

This is, in fact, what does occur. This definition, therefore, agrees with the earlier one.

Now, since (for this glass) λ, C, μ_0 are all positive, it follows that if we are to have a maximum followed by a minimum, as in fig. 10, $dQ/dT < 0$.

Now, in $Q = Ne^2\tau^2/m\pi$, since τ does not change, the only quantity which can change with the polarization is N, the number of electrons per unit volume.

dQ/dT is then $e^2\tau^2(N_2-N_1)/\pi m\,(T_2-T_1)$, where N_2, N_1 are the number of electrons of period τ which respond to vibrations in the directions of T_2 and T_1.

Our result will, therefore, signify that a tensional stress *decreases* the number of electrons which respond to vibrations in the direction of the stress, relatively to the number responding to vibrations in a perpendicular direction. In other words, tension appears to tend to set the electrons vibrating in a plane at right angles to the line of stress, pressure having the reverse effect.

It seems probable that if this effect were due to an absorption band, a glass

showing in its unstressed condition strong absorption bands in the visible spectrum would exhibit this effect in a most marked manner. A small slab of didymium glass was obtained and examined under direct pressure with the apparatus described in the previous paper in 'Camb. Phil. Soc. Proc.,' already referred to several times. The observations were not very precise, and it is hoped to repeat them with the flexure apparatus, when larger slabs can be obtained.* Such as they were, however, they gave a negative result. The band due to stress passed through the double absorption band in the orange without showing any marked irregularity.

It seems, therefore, that the didymium electrons which produce the absorption band are not affected by the stress in the way described. It appears possible that the didymium particles really float about, as it were, in the glass, in a free state, like particles in suspension in a fluid, and that they cannot be influenced to any great extent by stress applied to the glass. Further research in this direction is in progress.

§ 22. *Determination of Absolute Values of* C.

Although the experiments were primarily undertaken to show the dispersion effects, it is desirable to know also the absolute values of the stress-optical coefficients. These are not given by the experiments as described in § 2, because the differences of level $z_1 - z_2$ and $z_1 - h$ cannot be measured with sufficient accuracy.

To determine absolute values a second slit is used, of which the height is $h + \Delta h$. The two slits were cut in the same diaphragm, so that Δh is easily measured once for all.

Referring to formula (7), the relative retardation due to the second slit,

$$\mathrm{R}' = (3\mathrm{M}_2\mathrm{C}'_2/2b_2^3)\left[z_1 - z_2 + \sigma(z_1 - h - \Delta h)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (29),$$

where C'_2 = stress-optical coefficient of beam F for the wave-length for which the retardation is R′.

Also

$$\mathrm{R} = (3\mathrm{M}_2\mathrm{C}_2/2b_2^3)\left[z_1 - z_2 + \sigma(z_1 - h)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

Hence

$$\mathrm{R}'/\mathrm{C}'_2 - \mathrm{R}/\mathrm{C}_2 = -3\mathrm{M}_2\sigma\Delta h/2b_2^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

If therefore we know the ratio $\mathrm{C}'_2 : \mathrm{C}_2$ we can find the absolute values of either.

If R′, R correspond to bands of n^{th} order,

$$\mathrm{R} = n\lambda, \qquad \mathrm{R}' = n\lambda'.$$

Therefore

$$n(\lambda'/\mathrm{C}'_2 - \lambda/\mathrm{C}_2) = -3\mathrm{M}_2\sigma\Delta h/2b_2^3.$$

[**Note added April 3*, 1907.—Since writing the above the experiment has been repeated under the more accurate conditions, and the negative result has been confirmed. If the effect exists in the didymium glass it is certainly small.]

Assuming the law connecting C and λ to be given by (14), we have

$$C_2 = C_0^{(2)}/(1-\lambda_0^{(2)}/\lambda), \qquad C'_2 = C_0^{(2)}/(1-\lambda_0^{(2)}/\lambda'),$$

where $C_0^{(2)}$, $\lambda_0^{(2)}$ refer to the slab F.

Hence

$$n(\lambda'-\lambda)/C_0^{(2)} = -3M_2\sigma\Delta h/2b_2^3.$$

Therefore

$$C_0^{(2)} = -2n(\lambda'-\lambda)\,b_2^3/3M_2\sigma\Delta h \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

and from equation (5),

$$C_0^{(1)} = 2n(\lambda'-\lambda)\,b_1^3\left(\frac{1}{\sigma}+1\right)\Big/3M_1\Delta h \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

which give the absolute values of C_0 for both slabs.

A great many errors enter into the determination of these absolute values. It is very difficult to measure the spans with sufficient exactness, and the differences $\lambda'-\lambda$ are not large enough to allow of very accurate determination.

§ 23. *Effect of Chemical Composition on Stress-optical Properties.*

The mean values for C_0 obtained in this way for each beam are shown below in Table VI. A, B denote the two individual slabs of each pair, and C_0 is expressed in a unit equal to 10^{-7} (cm.)^2 per kilogramme weight.

TABLE VI.—Dependence of C_0, λ_0 on Chemical Composition.

Glass.	C_0 for A.	C_0 for B.	Mean C_0.	λ_0.	B_2O_3.	K_2O.	$B_2O_3 - \frac{1}{2}K_2O$.
3413	2·99	3·11	3·05	553	33·0	3·8	31·1
1809	2·95	2·94	2·94	561	34·3	7·4	30·6
935	2·82	2·94	2·88	451	27·7	3·1	26·1
3296	2·71	2·83	2·77	537	15·4	16·7	7·0
3749	2·15	2·19	2·17	564	5·9	23·9	− 6·1
3453	2·13	2·10	2·11	524	5·7	20·8	− 4·7
2783	1·93	2·21	1·93*	500†	1·4	12·5	− 4·9

* 1·93 has been taken, and not the mean of the two values, because here C_0 certainly differs for slabs A and B, and A was the slab analysed.

† Estimated from the values of Λ_r on p. 291.

If we refer to the diagram published in a paper previously referred to ('Camb. Phil. Soc. Proc.,' vol. XII., p. 335) showing dependence of C upon percentage of PbO, we see that until this percentage reaches about 40, PbO has little influence on the stress-optical effect.

We are therefore to look at the two remaining principal constituents, B_2O_3 and K_2O, for the cause of variations in C_0.

Looking at Table VI., in which the glasses are arranged according to descending order of magnitude of C_0, we see that the four glasses with the high C_0 all contain a percentage of B_2O_3 which is considerable. On the other hand, the three glasses with a low C_0 all contain a comparatively high percentage of K_2O. We conclude that either B_2O_3 raises C_0, or K_2O lowers it, or both. Also, looking at 2783, which is the lowest of the seven, we notice that it contains the least percentage of B_2O_3, but not the highest percentage of K_2O. This suggests that B_2O_3 is more efficient in raising C_0 than K_2O in lowering it. That the effect of B_2O_3 must be predominant is otherwise evident from the fact that the order of the percentages of B_2O_3 is the same as the order of magnitude of C_0, with one exception (1809).

The last column of Table VI. shows the values of (percentage of B_2O_3)$-\frac{1}{2}$ (percentage of K_2O). This places 1809 in the sequence, but throws out 3749. Also the marked difference between the four first and the three last is shown very clearly.

The glasses do not form a sufficiently regular series to enable us to go further and to determine exactly the law of dependence of C_0 upon the percentages of B_2O_3 and of K_2O in the glass. But the increase of B_2O_3 certainly increases C_0, and the increase of K_2O probably decreases it; and B_2O_3 seems to be at least twice as efficient as K_2O.

With regard to the mean values of λ_0, they all appear to be of the same order of magnitude. At all events no definite dependence of λ_0 upon the composition can be traced. For percentages of B_2O_3 not exceeding 35 and of K_2O not exceeding 25 it would seem that λ_0 is independent of the composition of the glass or that the dispersion is, in every case, proportional to the double refraction.

§ 24. *Failure of* HOOKE'S *Law for Glass* 2783.

Before concluding, the peculiar phenomena shown by glass 2783 require explanation.

This glass showed a progressive *increase* with increase of load in $r\Delta\lambda/\Delta W_r$ for both tension and pressure.

Also when the load was increased the band, which was straight and vertical for moderate loads, became curved, being convex towards the red as shown in fig. 12. The load was eventually increased to about 59 kilogrammes when one of the glasses broke. The band was observed when the glass was on the point of rupture and it exhibited a decided **V** shape as drawn.

This showed that the relative retardation was greater for light passing through at mid-level than for light passing through the edges of the beams.

Such an effect can be explained only in two ways. Firstly, by supposing that the law of stress across the section does indeed remain linear, but that the relative

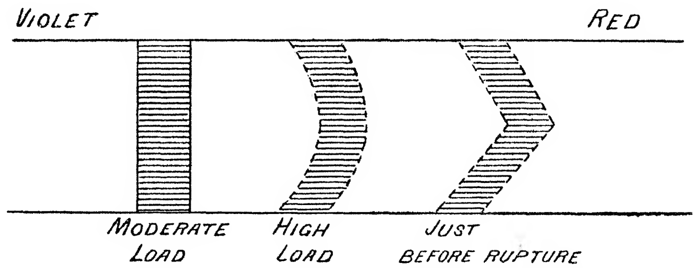

Fig. 12. Appearances of band for glasses 2783.

retardation is not proportional to the stress, increasing *less* rapidly for large stresses than for small, since here (the overlapping being assumed for simplicity to be $\frac{1}{2}$ height of beams) $\frac{1}{2}T+\frac{1}{2}T$ produces more effect than T.

Such an effect, however, would imply an increasing falling off of the values of λ from the values which would be obtained if the linear law held. The result would be a progressive *decrease*, instead of a progressive *increase*, in $r\Delta\lambda/\Delta W_r$.

Secondly, we may suppose that HOOKE'S law fails or stress is not proportional to strain. Consider a beam bent under constant bending moment. The axis will take the form of a circular arc. If we imagine the circle completed, then the symmetry shows that the cross-sections must remain plane and all pass through the centre of the circle.

It follows easily that, *whatever be the law of stress*, if we can *neglect end effects*, the extension follows a strictly linear law.

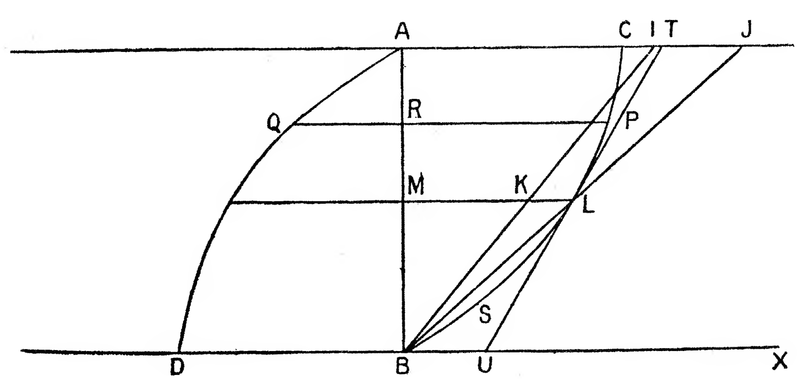

Fig. 13.

Thus if AB (fig. 13) represent the vertical axis of such a section, and the stress at any point R in AB be set off as RP perpendicular to AB, the locus BPC of P is a

genuine stress-strain diagram. It will therefore, as is well known, take the shape shown in fig. 13, the stress falling off rapidly as the strain approaches and passes what is known as the yield-point.

The corresponding curve for the overlapping part of the other beam is shown by AQD, and the stress effective in producing the optical effect is the sum of the two ordinates RP, RQ. It is obvious from the figure that PQ is a maximum in the middle. Thus the peculiar shape of the band is likewise accounted for on this hypothesis.

Now let the straight line BI give the stress-diagram which would be obtained for the same bending moment if HOOKE'S law held. This straight line and the curve must then be so related that the first moment about BX of the areas APCB and AIB are equal.

Let L be the extremity of the ordinate through the mid-point of AB. Draw BL cutting AC at J and the tangent at L cutting AC at T, BX at U. The stress-strain curve is always convex inwards, therefore CPLB always lies on one side of TU as shown.

Now the triangles BLU, TLJ are clearly equal. Hence area CPLJ $>$ area BSL. And the mean distance from BX of the area CPLJ $>$ mean distance from BX of the area BSL. Therefore first moment of CPLJ $>$ first moment of BSL, or first moment of ABJ $>$ first moment of ABLC. Therefore, if ABI and ABLC have the same first moment, ABI $<$ ABJ, or BI must lie to the left of BJ.

If BI cut ML in K, then ML $>$ MK. That is, the actual measured stress is greater than the computed stress. Therefore the observed values of λ exceed those that would be obtained if HOOKE'S law held by a difference rapidly increasing with the stress.

The result is a progressive increase in the value of $r\Delta\lambda/\Delta W_r$ such as is actually observed. The discrepancies which have appeared are therefore explained.

Incidentally this confirms the conclusion (which indeed seems highly probable on theoretical grounds) that the stress-optical effect is dependent upon the stress—that is, the molecular strain—and not upon the molar strain. The latter, which is the sum of both plastic and elastic effects, is the quantity measured in most extension experiments, and is usually denoted by "strain" simply.

§ 25. *Conclusion.*

This completes the account of the results reached so far. The next step would be to obtain glasses of suitable chemical composition to show the effects discovered in a much stronger degree and thus allow of more precise determinations. Research in this direction is being undertaken, and it is hoped that the results will form the subject of a future communication.

In conclusion the author wishes to express his most grateful thanks for the assistance which has been rendered him by the Royal Society, without which this work could not have been undertaken. He desires also to place on record his indebtedness to Professor F. T. Trouton, F.R.S., who has most generously placed the resources of the Physical Laboratory of University College, London, at his disposal; and also to Mr. A. W. Porter, B.Sc., whose interest in the work and kindly criticisms and suggestions have been invaluable.